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SOME EXTENSIONS OF THE WISHART DISTRIBUTION[1]

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Princeton University and Columbia University

1. Introduction. The well-known Wishart distribution is the distribution of the variances and covariances of a sample drawn from a multivariate normal population assuming that the expected value of each variate remains the same from observation to observation. For problems such as testing collinearity [1], comparing scales of measurement [2], and multiple regression in times series analysis [3], it is desirable to have the distribution of sample variances and covariances for observations, the expected values of which are not all identical. Such a distribution could be considered as a generalization to several variates of the χ'^2 (non-central χ^2) distribution, as well as a generalization of the Wishart distribution to the non-central case. In this paper we shall discuss the general problem of finding the distribution in question and shall derive this distribution for two particular cases. We shall start out with the problem in its most general form and as a result of linear transformations express the distribution as a certain multiple integral.

We can think of the expected values of the observations as defining points in a space of dimension equal to the number of variates. If these points lie on a line, the non-central Wishart distribution is essentially the Wishart distribution multiplied by a Bessel function; if the points lie in a plane, it is a Wishart distribution multiplied by an infinite series of Bessel functions. For higher dimensionality the integration of the multiple integral becomes extremely troublesome; it has not been possible yet to express the general integration in a concise form. These results are summarized precisely at the end of the paper.

2. Reduction to canonical form. Consider a set of N multivariate normal populations each of p variates. Let the ith ($i = 1, 2, \cdots, p$) variate of the αth ($\alpha = 1, 2, \cdots, N$) population be $x_{i\alpha}$; let the mean of this variate be

$$E(x_{i\alpha}) = \mu_{i\alpha} \quad (i = 1, 2, \cdots, p;\ \alpha = 1, 2, \cdots, N); \tag{1}$$

and let the variance-covariance matrix (of rank p) common to all N distributions be

$$\| E[(x_{i\alpha} - \mu_{i\alpha})(x_{j\alpha} - \mu_{j\alpha})] \| = \| \sigma_{ij} \| \quad (\alpha = 1, 2, \cdots, N).$$

Now consider a sample of observations $\{x_{i\alpha}\}$ one from each population.

The purpose of this paper is to find the joint distribution of the quantities

$$a_{ij} = \sum_{\alpha=1}^{N} (x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j), \tag{2}$$

[1] The results given below were arrived at independently by the two authors. Some preliminary results were given before the Institute of Mathematical Statistics at Washington, D. C., May 6, 1944, by Girshick.

where

$$\bar{x}_i = \frac{1}{N}\sum_{\alpha=1}^{N} x_{i\alpha} .$$

To simplify the notation in the subsequent work we treat the quantities a_{ij} instead of the sample variances and covariances which are simply multiples (by $1/(N-1)$) of the a_{ij}.

The a_{ij} may be considered as sums of squares and cross products, for there exists a linear transformation[2].

$$x'_{i\alpha} = \sum_{\beta=1}^{N} \theta_{\alpha\beta} x_{i\beta} \qquad (i = 1, 2, \cdots, p),$$

where the matrix $\| \theta_{\alpha\beta} \|$ is orthogonal (and $\theta_{N1} = \theta_{N2} = \cdots = \theta_{NN} = 1/\sqrt{N}$), such that

$$a_{ij} = \sum_{\alpha=1}^{n} x'_{i\alpha} x'_{j\alpha} ,$$

where $n = N - 1$ and

$$N\bar{x}_i\bar{x}_j = x'_{iN}x'_{jN} .$$

For a given α the $x'_{i\alpha}$ have a multivariate normal distribution with the same variances and covariances as the x's and with expected values

$$E(x'_{i\alpha}) = \sum_{\beta=1}^{N} \theta_{\alpha\beta}\mu_{i\beta} = \mu'_{i\alpha} , \quad \text{say.}$$

Let

$$\tau_{ij} = \sum_{\alpha=1}^{n} \mu'_{i\alpha}\mu'_{j\alpha} .$$

Then it is clear that the τ_{ij} are the same functions of the μ's that the a_{ij} are of the x's, namely,

$$\tau_{ij} = \sum_{\alpha=1}^{N} (\mu_{i\alpha} - \bar{\mu}_i)(\mu_{j\alpha} - \bar{\mu}_j), \tag{3}$$

where

$$\bar{\mu}_i = \frac{1}{N}\sum_{\alpha=1}^{N} \mu_{i\alpha} .$$

Now consider the two p by p matrices

$$\Sigma = \| \sigma_{ij} \|$$

and

$$\mathrm{T} = \| \tau_{ij} \|.$$

---

[2] See, for example, [4].

Let $\kappa_1^2, \kappa_2^2, \cdots, \kappa_p^2$ be the real, non-negative roots of the determinantal equation

$$| \mathrm{T} - \lambda\Sigma | = 0. \tag{4}$$

There exists a non-singular p by p matrix

$$\Psi = \| \psi_{ij} \| \tag{5}$$

such that[3]

$$\Psi\Sigma\Psi' = \mathrm{I} \tag{6}$$

and

$$\Psi\mathrm{T}\Psi' = \begin{Vmatrix} \kappa_1^2 & 0 & \cdots & 0 \\ 0 & \kappa_2^2 & \cdots & 0 \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ 0 & 0 & \cdots & \kappa_p^2 \end{Vmatrix}, \tag{7}$$

where I is the identity matrix and Ψ' is the transpose of Ψ. Suppose the rank of T is t; then t of the roots are non-zero and $p - t$ are zero. For the sake of convenience we shall choose $\kappa_1^2, \kappa_2^2, \cdots, \kappa_t^2$ to be the non-zero roots. If T is of rank t, then the means $\mu_{i\alpha}$ lie in a t dimensional sub-space of the original p dimensional space. Let us make the transformation

$$z_{i\alpha} = \sum_{j=1}^{p} \psi_{ij} x'_{j\alpha}. \tag{8}$$

The $z_{i\alpha}$ are normally and, because of relationship (6), independently distributed with unit variances. The mean value of $z_{i\alpha}$ is

$$E(z_{i\alpha}) = \sum_{j=1}^{p} \psi_{ij} \mu'_{j\alpha} = \nu_{i\alpha},$$

say. As a result of (7)

$$\sum_{\alpha=1}^{n} \nu_{i\alpha}^2 = \kappa_i^2 \qquad (i = 1, 2, \cdots, p), \tag{9}$$

$$\sum_{\alpha=1}^{n} \nu_{i\alpha} \nu_{j\alpha} = 0 \qquad (i \neq j). \tag{10}$$

Let the new sum of squares of cross-products be

$$b_{ij} = \sum_{\alpha=1}^{n} z_{i\alpha} z_{j\alpha}. \tag{11}$$

We shall first find the joint distribution of the b_{ij} and then obtain the distribution of the a_{ij} by using the fact that the b_{ij} can be considered simply as a linear

[3] See, for example, [5].

transformation of the a_{ij} in a $\frac{1}{2}p(p+1)$ dimensional space. For we can write b_{ij} as

$$b_{ij} = \sum_{\alpha=1}^{n} \sum_{h,k=1}^{p} \psi_{ih}\psi_{jk} x'_{h\alpha} x'_{k\alpha} = \sum_{h,k=1}^{p} \psi_{ih}\psi_{jk} a_{hk} . \tag{12}$$

The transformation (8) is performed on all variates of each observation. The next transformation, which is the one that results in the canonical form of the problem, is performed on all observations of each variate. We wish to construct the n by n matrix of this transformation

$$\Phi = \| \phi_{\alpha\beta} \|$$

in the following manner: Let

$$\phi_{\eta\alpha} = \frac{\nu_{\eta\alpha}}{\kappa_\eta} \qquad (\alpha = 1, 2, \cdots, n; \eta = 1, 2, \cdots, t).$$

In view of (9) and (10)

$$\sum_{\alpha=1}^{n} \phi_{\xi\alpha}\phi_{\eta\alpha} = \delta_{\xi\eta} , \qquad (\xi, \eta = 1, 2, \cdots, t)$$

where $\delta_{\xi\eta}$ is the Kronecker delta. The remaining elements in Φ are chosen in any way to make Φ orthogonal.

Now make the transformation

$$y_{i\alpha} = \sum_{\beta=1}^{n} \phi_{\alpha\beta} z_{i\beta} \qquad (i = 1, 2, \cdots, p; \alpha = 1, 2, \cdots, n).$$

Because Φ is orthogonal,

$$b_{ij} = \sum_{\alpha=1}^{n} y_{i\alpha} y_{j\alpha} , \tag{13}$$

and the y's are independently normally distributed. By virtue of the construction of Φ and the properties of $\| \nu_{i\alpha} \|$ the expected value of each $y_{i\alpha}$ is zero except for t of the variates, namely,

$$E(y_{\eta\eta}) = \kappa_\eta \qquad (\eta = 1, 2, \cdots, t).$$

Now the problem can be put in this form: Find the distribution of b_{ij} (given by (13)) when the distribution of the y's (in the canonical form) is

$$\frac{1}{(2\pi)^{pn/2}} e^{-\frac{1}{2}\sum_{i=1}^{p}\sum_{\alpha=1}^{n}(y_{i\alpha}-\kappa_i\delta_{i\alpha})^2} ,$$

where $\kappa_1, \kappa_2, \cdots, \kappa_t$ are different from zero.

The solution of our problem can be expressed as a certain multiple integral of tp variables. Let

$$b'_{ij} = \sum_{\alpha=t+1}^{n} y_{i\alpha} y_{j\alpha} .$$

Since the b'_{ij} have the Wishart distribution with $n - t$ degrees of freedom (we assume $n \geq t + p$) we can write the joint distribution of the b'_{ij} and $y_{i\eta}$ $(i = 1, 2, \cdots, p; \eta = 1, 2, \cdots, t)$ as

$$\frac{1}{2^{\frac{1}{2}p(n-t)}\pi^{\frac{1}{2}p(p-1)}\prod_{i=1}^{p}\Gamma(\frac{1}{2}[n-t+1-i])}\,|b'_{ij}|^{\frac{1}{2}(n-p-t-1)}e^{-\frac{1}{2}\sum_{i,j=1}^{p}b'_{ij}} \times \frac{1}{(2\pi)^{pt/2}}e^{-\frac{1}{2}\sum_{i=1}^{p}\sum_{\eta=1}^{t}(y_{i\eta}-\kappa_\eta\delta_{i\eta})^2}.$$

Considering the equations

$$b_{ij} = b'_{ij} + \sum_{\eta=1}^{t} y_{i\eta}y_{j\eta}$$

as a transformation of the b'_{ij} we immediately obtain the joint distribution of the b's and the $y_{i\eta}(i = 1, 2, \cdots, p; \eta = 1, 2, \cdots, t)$ as

$$(14)\quad \frac{e^{-\frac{1}{2}\sum_{\eta=1}^{t}\kappa_\eta^2}}{2^{\frac{1}{2}pn}\pi^{\frac{1}{2}p(p-1)+\frac{1}{2}pt}\prod_{i=1}^{p}\Gamma(\frac{1}{2}[n-t+1-i])}\,|b_{ij}-\sum_{\eta=1}^{t}y_{i\eta}y_{j\eta}|^{\frac{1}{2}(n-p-t-1)} \times e^{-\frac{1}{2}\sum_{i=1}^{p}b_{ii}+\sum_{\eta=1}^{t}\kappa_\eta y_{\eta\eta}}.$$

To find the distribution of the b_{ij} we must integrate out the $y_{i\eta}$, where the range of integration is such that the matrix

$$\| b_{ij} - \sum_{\eta=1}^{t} y_{i\eta}y_{j\eta} \|$$

is positive. For $t = 1$ or 2 we can integrate (14) and express the results in a convenient form. However, for higher values of t the integration affords considerable difficulty and has not been done for the general case. In terms of geometry the case $t = 1$ is the case in which the expected values of the observations lie on a line in the p dimensional space. In the case of $t = 2$, similarly the expected values lie in a plane in this space. Hence, we shall call these two cases the linear and planar cases, respectively.

3. The linear case. In the linear case there is one root of the equation (4) which is not equal to zero, that is, there is simply one κ in the distribution (14) and one set of y's, namely $y_{i1}(i = 1, 2, \cdots, p)$. The problem is to integrate the y_{i1} over the range for which the matrix

$$\| b_{ij} - y_{i1}y_{j1} \|$$

is positive; the integrand we are interested in is (dropping the subscript "1" from the κ_1 and y_{i1} and neglecting the part not involving the y's)

$$|\, b_{ij} - y_i y_j \,|^{\frac{1}{2}(n-p-2)} e^{\kappa y_1} \prod_{i=1}^{p} dy_i \,. \tag{15}$$

The determinant in expression (15) can be expressed as

$$|\, b_{ij} - y_i y_j \,| = |\, b_{ij} \,| \left(1 - \sum_{i,j=1}^{p} b^{ij} y_i y_j\right),$$

where

$$\| b^{ij} \| = \| b_{ij} \|^{-1}.$$

The inverse exists because the probability is zero that $\| b_{ij} \|$ is singular. There is a linear transformation

$$y_i = \sum_{j=1}^{p} q_{ij} u_j \,,$$

such that

$$\sum_{i,j=1}^{p} b^{ij} y_i y_j = \sum_{j=1}^{p} u_j^2$$

and

$$\kappa y_1 = l u_1 \,,$$

where l^2 is the one non-zero root of the equation:

$$|\, \lambda B^{-1} - \kappa^2 E_{11} \,| = 0, \tag{16}$$

where $B^{-1} = \| b^{ij} \|$ and E_{11} is the matrix with unity in the upper left hand corner and zeros elsewhere. This fact is a result of the well-known theorem concerning diagonalization of pairs of quadratic forms.[4] The Jacobian of this transformation is

$$|\, q_{ij} \,| = |\, b_{ij} \,|^{\frac{1}{2}},$$

and the range of integration is $\sum_{i=1}^{p} u_i^2 \leq 1$. The integrand is transformed into

$$|\, b_{ij} \,|^{\frac{1}{2}(n-p-1)} \cdot \left(1 - \sum_{i=1}^{p} u_i^2\right)^{\frac{1}{2}(n-p-2)} e^{l u_1} \prod_{i=1}^{p} du_i \,.$$

Now let

$$u_1 = \sin w,$$

$$u_i = \cos w \, v_{i-1} \qquad (i = 2, 3, \cdots, p)$$

[4] The transformation is the so-called "regression transformation." See Madow [6].

The Jacobian of this transformation is $\cos^p w$ and

$$1 - \sum_{i=1}^{p} u_i^2 = \cos^2 w \left(1 - \sum_{i=1}^{p-1} v_i^2\right).$$

The integration is over the ranges of $\frac{\pi}{2} \leq w \leq \frac{\pi}{2}$ and

$$\sum_{i=1}^{p-1} v_i^2 \leq 1.$$

We integrate the following expression

$$|b_{ij}|^{\frac{1}{2}(n-p-1)} \left[\left(1 - \sum_{i=1}^{p-1} v_i^2\right)^{\frac{1}{2}(n-p-2)} \prod_{i=1}^{p-1} dv_i\right] \{\cos^{n-2} w\, e^{l \sin w}\, dw\}.$$

The integral of the quantity within the brackets is simply a Dirichlet integral [7] and its value is

$$\frac{\Gamma(\frac{1}{2}[n-p])\pi^{\frac{1}{2}(p-1)}}{\Gamma(\frac{1}{2}[n-1])}.$$

The integral of the expression within the braces is a multiple of a Bessel function of purely imaginary argument [8, p. 79]; that is,

$$\frac{\Gamma(\frac{1}{2}[n-1])\sqrt{\pi}}{(l/2)^{\frac{1}{2}(n-2)}} I_{\frac{1}{2}(n-2)}(l).$$

Hence, the integral of (15) is

$$|b_{ij}|^{\frac{1}{2}(n-p-1)} \Gamma(\tfrac{1}{2}[n-p])\pi^{\frac{1}{2}p}(l/2)^{-\frac{1}{2}(n-2)} I_{\frac{1}{2}(n-2)}(l). \tag{17}$$

Multiplying equation (16) by the determinant $|B|$ one can easily show that the non-zero root, l^2, is simply $\kappa^2 b_{11}$. The distribution of the b_{ij} in the linear case then is

$$\frac{e^{-\frac{1}{2}\kappa^2} e^{-\frac{1}{2}\sum_{i=1}^{p} b_{ii}}}{2^{\frac{1}{2}pn - \frac{1}{2}(n-2)} \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p-1} \Gamma(\frac{1}{2}[n-i])} |b_{ij}|^{\frac{1}{2}(n-p-1)} (\kappa^2 b_{11})^{-\frac{1}{2}(n-2)} I_{\frac{1}{2}(n-2)}(\kappa\sqrt{b_{11}}),$$

In §5 we shall give the distribution in terms of the original variables, namely, the a_{ij}.

4. The planar case. The case of two non-zero roots of equation (4) can be handled by continuing the process of integration of §3 another step. The essential problem is the integration of

$$|b_{ij} - \sum_{\eta=1}^{2} y_{i\eta} y_{j\eta}|^{\frac{1}{2}(n-p-3)} e^{\kappa_1 y_{11} + \kappa_2 y_{22}} \prod_{i=1}^{p} \prod_{\eta=1}^{2} dy_{i\eta} \tag{18}$$

over the range of y's for which the matrix

$$|| b_{ij} - \sum_{\eta=1}^{2} y_{i\eta} y_{j\eta} ||$$

is positive. The integration is done in two stages, first with respect to the y_{i1}, then with respect to the y_{i2}. Letting

$$\begin{aligned} \tilde{b}_{ij} &= b_{ij} - y_{2i} y_{2j} , \\ \tilde{n} &= n - 1, \\ \tilde{\kappa} &= \kappa_1 , \\ \tilde{y}_i &= y_{i1} , \end{aligned}$$

and omitting for the time being

$$e^{\kappa_2 y_{22}} \prod_{i=1}^{p} dy_{i2} , \tag{19}$$

we can write the first stage of the integration of (18) as

$$\int | \tilde{b}_{ij} - \tilde{y}_i \tilde{y}_j |^{\frac{1}{2}(\tilde{n}-p-2)} e^{\tilde{\kappa}\tilde{y}_1} \prod_{i=1}^{p} d\tilde{y}_i$$

over the range $|| \tilde{b}_{ij} - \tilde{y}_i \tilde{y}_j ||$ positive. But, the only difference between this and the integration of (15) which has been shown to be (17), is that we are now writing all variables with "$\sim$" signs. Keeping this in mind, changing back again to our other variables of Section 4 and inserting again (19) we can write the first stage of the integration of (18) as

$$\begin{aligned} &\Gamma(\tfrac{1}{2}[n-p-1])\pi^{\frac{1}{2}p} \, | b_{ij} - y_{i2} y_{j2} |^{\frac{1}{2}(n-p-2)} e^{\kappa_2 y_{22}} \left[\frac{\kappa_1^2 (b_{11} - y_{12}^2)}{4} \right]^{-\frac{1}{2}(n-3)} \\ &\qquad \cdot I_{\frac{1}{2}(n-3)} (\sqrt{\kappa_1^2 b_{11} - \kappa_1^2 y_{12}^2}) \prod_{i=1}^{p} dy_{i2} . \end{aligned} \tag{20}$$

Now we must integrate (20) with respect to the y_{i2} over the range $|| b_{ij} - y_{i2} y_{j2} ||$ positive. The determinant in (20) can be written as

$$| b_{ij} | \left(1 - \sum_{i,j=1}^{p} b^{ij} y_{i2} y_{j2} \right).$$

There is a transformation

$$y_{i2} = \sum_{j=1}^{p} g_{ij} s_j , \tag{21}$$

such that

$$\begin{aligned} \sum_{i,j=1}^{p} b^{ij} y_{i2} y_{j2} &= \sum_{j=1}^{p} s_j^2 , \\ \kappa_1^2 y_{12}^2 &= f^2 s_1^2 , \\ \kappa_2 y_{22} &= d_1 s_1 + d_2 s_2 , \end{aligned} \tag{22}$$

where f^2 is the one non-zero root of the equation

$$|\lambda B^{-1} - \kappa_1^2 E_{11}| = 0,$$

where B^{-1} and E_{11} are used as in equation (16). Since this equation is similar to (14), then $f^2 = \kappa_1^2 b_{11}$. The values of d_1 and d_2 will be considered later. This result is deduced from an extension of the theorem concerning the diagonalization of pairs of quadratic forms.[5] The Jacobian is

$$|g_{ij}| = |b_{ij}|^{\frac{1}{2}},$$

and the range of integration is $\sum_{i=1}^{p} s_i^2 \leq 1$. The integrand (20) is now changed to

$$\Gamma(\tfrac{1}{2}[n-p-1])\pi^{\frac{1}{2}p}|b_{ij}|^{\frac{1}{2}(n-p-1)}\left(1-\sum_{i=1}^{p}s_i^2\right)^{\frac{1}{2}(n-p-2)}e^{d_1 s_1+d_2 s_2}$$
$$\cdot\left[\frac{f^2(1-s_1^2)}{4}\right]^{-\frac{1}{2}(n-3)} I_{\frac{1}{2}(n-3)}\left(\sqrt{f^2(1-s_1^2)}\right)\prod_{i=1}^{p} ds_i .$$

Next the following transformation is made:

$$s_1 = \sin w_1 ,$$
$$s_2 = \cos w_1 \sin w_2 ,$$
$$s_i = \cos w_1 \cos w_2 v_{i-2} \qquad (i = 3, 4, \cdots, p).$$

The Jacobian is $\cos^p w_1 \cos^{p-1} w_2$, and

$$1 - \sum s_i^2 = \cos^2 w_1 \cos^2 w_2 \left(1 - \sum_{i=1}^{p-2} v_i^2\right).$$

We now integrate

$$\Gamma(\tfrac{1}{2}[n-p-1])\pi^{\frac{1}{2}p}|b_{ij}|^{\frac{1}{2}(n-p-1)}\left[\left(1-\sum_{i=1}^{p-2}v_i^2\right)^{\frac{1}{2}(n-p-2)}\prod_{i=1}^{p-2}dv_i\right]$$

(23)
$$\cdot\left\{\cos^{n-2} w_1 \cos^{n-3} w_2 e^{d_1 \sin w_1 + d_2 \cos w_1 \sin w_2}\left(\frac{f}{2}\cos w_1\right)^{-\frac{1}{2}(n-3)}\right.$$
$$\left.\cdot I_{\frac{1}{2}(n-3)}(f\cos w_1)\, dw_2\, dw_3\right\}.$$

The integral of the expression within the square brackets is another Dirichlet integral; its value is

$$\frac{\Gamma(\tfrac{1}{2}[n-p])\pi^{\frac{1}{2}(p-2)}}{\Gamma(\tfrac{1}{2}[n-2])}.$$

---

[5] Again the "regression transformation" is used. See footnote for Section 3.

Similar to Section 3 the integration of the quantity within the braces with respect to w_2 is

$$\Gamma(\tfrac{1}{2}[n-2])\sqrt{\pi}\left(\frac{d_2\cos w_1}{2}\right)^{-\frac{1}{2}(n-3)} I_{\frac{1}{2}(n-3)}(d_2\cos w_1)$$
$$\cdot\cos^{n-2} w_1 e^{d_1\sin w_1}\left(\frac{f}{2}\cos w_1\right)^{-\frac{1}{2}(n-3)} I_{\frac{1}{2}(n-3)}(f\cos w_1).$$

Since the range of integration of w_1 is $-\pi/2 \leq w_1 \leq \pi/2$ and since $\sin w_1$ is an odd function and $\cos w_1$ is an even function, the integral of the above expression can be transformed into an integration over the range $0 \leq w_1 \leq \pi/2$ by replacing $e^{d_1\sin w_1}$ by $2\sinh(d_1\sin w_1)$. In view of the relationship between the Bessel functions of purely imaginary argument and $\sinh(d_1\sin w_1)$ [8, p. 54] we can write the integral of the above expression as

$$\Gamma(\tfrac{1}{2}[n-2])\sqrt{\pi}\left(\frac{d_2}{2}\right)^{-\frac{1}{2}(n-3)}\sqrt{d_1}\sqrt{2\pi}$$
$$\times\int_0^{\pi/2} I_{-\frac{1}{2}}(d_1\sin w_1)I_{\frac{1}{2}(n-3)}(d_2\cos w_1)I_{\frac{1}{2}(n-3)}(f\cos w_1)\sin\tfrac{1}{2}w_1\cos w_1\,dw_1.$$

This integral can be expressed in another form by virtue of a formula in Watson's *Bessel Functions* [8, p. 377] as

$$\sqrt{\pi}\,2^{\frac{1}{2}(n-2)}\int_0^{\pi}\frac{I_{\frac{1}{2}(n-2)}(\sqrt{d_1^2+d_2^2+f^2-2d_2f\cos u})}{(d_1^2+d_2^2+f^2-2d_2f\cos u)^{\frac{1}{2}(n-2)}}\sin^{n-3}u\,du. \tag{24}$$

Letting $d_1^2+d_2^2+f^2=x$ and $d_2f=y$ and using an expresion formula for Bessel functions [8, p. 140] we can write (24) as

$$\sqrt{\pi}\,2^{\frac{1}{2}(n-2)}\int_0^{\pi}\sum_{\gamma=0}^{\infty}\frac{(-y)^{\gamma}\cos^{\gamma}u}{\gamma!}x^{-\frac{1}{2}(n-2)+\gamma}I_{\frac{1}{2}(n-2)+\gamma}(\sqrt{x})\sin^{n-3}u\,du.$$

Since the integral of $\cos^{\gamma}u\sin^{n-3}u$ where γ is odd is zero, the result of the integration (using the "duplication formula" for Γ functions and letting $\gamma=2\omega$) is

$$\Gamma(\tfrac{1}{2}[n-2])\pi 2^{\frac{1}{2}(n-2)}\sum_{\omega=0}^{\infty}\frac{(y^2)^{\omega}x^{-\frac{1}{2}(n-2)+2\omega}}{2^{2\omega}\omega!\Gamma\left(\frac{n-1}{2}+\omega\right)}I_{\frac{1}{2}(n-2)+2\omega}(\sqrt{x}).$$

From the relationship (22) it is clear that the equation

$$|\,\kappa_i^2\delta_{ij}-\lambda b^{ij}\,|=0 \quad (\kappa_i^2=0 \text{ for } i=3,4,\cdots,p) \tag{25}$$

is transformed by (21) into

$$\begin{vmatrix} f^2+d_1^2-\lambda & d_1d_2 & 0 & \cdots & 0\\ d_1d_2 & d_2^2-\lambda & 0 & \cdots & 0\\ 0 & 0 & -\lambda & \cdots & 0\\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ 0 & 0 & 0 & \cdots & -\lambda \end{vmatrix}=0. \tag{26}$$

Expression (25) is equivalent to

$$| k_i k_j b_{ij} - \lambda \delta_{ij} | = 0. \tag{27}$$

Hence, x and y^2, which are the sum and product, respectively, of the non-zero roots of the two equivalent equations (26) and (27), are given by

$$x = f^2 + d_1^2 + d_2^2 = \kappa_1^2 b_{11} + \kappa_2^2 b_{22},$$
$$y^2 = f^2 d_2^2 = \kappa_1^2 \kappa_2^2 (b_{11} b_{22} - b_{12}^2).$$

In view of this result we can now write the integration of (23) as

$$\Gamma(\tfrac{1}{2}[n-p-1])\Gamma(\tfrac{1}{2}[n-p])\pi^p 2^{\frac{1}{2}(n-2)} | b_{ij} |^{\frac{1}{2}(n-p-1)} \times \sum_{\omega=0}^{\infty} \frac{[\kappa_1^2 \kappa_2^2 (b_{11} b_{22} - b_{12}^2)]^\omega}{2^{2\omega} \omega! \Gamma\left(\frac{n-1}{2} + \omega\right)}$$
$$\cdot (\kappa_1^2 b_{11} + \kappa_2^2 b_{22})^{-\frac{1}{2}(n-2/2+2\omega)} I_{\frac{1}{2}(n-2)+2\omega}(\sqrt{\kappa_1^2 b_{11} + \kappa_2^2 b_{22}}).$$

Finally, by multiplying in what was left out of (18) we obtain the integral of (14) which is the solution to the problem as stated in the canonical form:

$$\frac{e^{-\frac{1}{2}(\kappa_1^2+\kappa_2^2)} | b_{ij} |^{\frac{1}{2}(n-p-1)} e^{-\frac{1}{2}\sum_{i=1}^{p} b_{ii}}}{2^{\frac{1}{2}pn-\frac{1}{2}(n-2)} \pi^{\frac{1}{2}p(p-1)} \prod_{i=1}^{p-2} \Gamma(\tfrac{1}{2}[n-1-i])} \tag{28}$$
$$\times \sum_{\omega=0}^{\infty} \frac{[\kappa_1^2 \kappa_2^2 (b_{11} b_{22} - b_{12}^2)]^\omega}{2^{2\omega} \omega! \Gamma\left(\frac{n-1}{2} + \omega\right)} (\kappa_1^2 b_{11} + \kappa_2^2 b_{22})^{-\frac{1}{2}(\frac{1}{2}(n-2)+2\omega)}$$
$$\times I_{\frac{1}{2}(n-2)+2\omega}(\sqrt{\kappa_1^2 b_{11} + \kappa_2^2 b_{22}}).$$

5. Final form. To answer the problem as stated originally it is necessary to make the transformation (12) and obtain the distribution in terms of the a_{ij} for the linear and planar cases.

It is clear that equation (25) is equivalent to

$$| T - \lambda A^{-1} | = 0, \tag{29}$$

where $A^{-1} = || a_{ij} ||^{-1}$, for T is the transform (by (5)) of $|| \kappa_i^2 \delta_{ij} ||$ and A^{-1} is the transform of $|| b_{ij} ||$. The sum and product of the non-zero roots, which are the arguments of the infinite series in (24), remain unchanged.

Since the quantity $\sum_{i=1}^{p} \kappa_i^2$ is the sum of the roots of (4) it can be expressed as

$$\sum_{i=1}^{p} \kappa_i^2 = \sum_{i,j=1}^{p} \sum_{\alpha=1}^{N} \sigma^{ij} (\mu_{i\alpha} - \bar{\mu}_i)(\mu_{j\alpha} - \bar{\mu}_j),$$

where $|| \sigma^{ij} || = || \sigma_{ij} ||^{-1}$. Furthermore we have

$$| b_{ij} | = | \Sigma | \cdot | a_{ij} |$$

and

$$\sum_{i=1} b_{ii} = \operatorname{tr} \| b_{ij} \| = \operatorname{tr} (\Psi \| a_{ij} \| \Psi') = \sum_{i,j=1}^{p} \sigma^{ij} a_{ij}.$$

Moreover, the Jacobian of the transformation (12) is[6]

$$| J | = | \Psi |^{p+1} = | \Sigma |^{-\frac{1}{2}(p+1)}. \tag{30}$$

Hence, we have the following results:

Given N multivariate normal populations each of p variates with identical variance-covariance matrices $\| \sigma_{ij} \|$ and with expected values of the pN variates $x_{i\alpha}$ given by (1). *Let a_{ij} be defined by equation* (2); *let the rank of the matrix $\| \tau_{ij} \|$ defined by* (3) *be t.*

(i) *When $t = 0$, the joint distribution of the a_{ij} is given by the Wishart distribution.*

(ii) *When $t = 1$, the joint distribution of the a_{ij} is*

$$\frac{e^{-\frac{1}{2}\sum_{i,j=1}^{p}\sum_{\alpha=1}^{N}\sigma^{ij}(\mu_{i\alpha}-\bar{\mu}_i)(\mu_{j\alpha}-\bar{\mu}_j)} | \sigma^{ij} |^{\frac{1}{2}(N-1)} | a_{ij} |^{\frac{1}{2}(N-p-2)}}{2^{\frac{1}{2}p(N-1)-\frac{1}{2}(N-3)}\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p-1}\Gamma(\frac{1}{2}[N-1-i])}$$

$$\times e^{\frac{1}{2}\sum_{i,j=1}^{k}\sigma^{ij}a_{ij}}\left[\sum_{i,j=1}^{p}\sum_{\alpha=1}^{N}a_{ij}(\mu_{i\alpha}-\bar{\mu}_i)(\mu_{j\alpha}-\bar{\mu}_j)\right]^{-\frac{1}{2}(N-3)} \tag{31}$$

$$\times I_{\frac{1}{2}(N-3)}\left(\sqrt{\sum_{i,j=1}^{p}\sum_{\alpha=1}^{N}a_{ij}(\mu_{i\alpha}-\bar{\mu}_i)(\mu_{j\alpha}-\bar{\mu}_j)}\right).$$

(iii) *When $t = 2$, the joint distribution of the a_{ij} is given by*

$$\frac{e^{-\frac{1}{2}\sum_{i,j=1}^{p}\sum_{\alpha=1}^{N}\sigma^{ij}(\mu_{i\alpha}-\bar{\mu}_i)(\mu_{j\alpha}-\bar{\mu}_j)} | \sigma^{ij} |^{\frac{1}{2}(N-1)} | a_{ij} |^{\frac{1}{2}(N-p-2)}}{2^{\frac{1}{2}p(N-1)-\frac{1}{2}(N-3)}\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p-2}\Gamma(\frac{1}{2}[N-2-i])}$$

$$\times e^{-\frac{1}{2}\sum_{i,j=1}^{p}\sigma^{ij}a_{ij}} \times \sum_{\omega=0}^{\infty}\frac{(u_1u_2)^{\omega}}{2^{2\omega}\omega!\,\Gamma\left(\frac{N-2}{2}+\omega\right)}(u_1+u_2)^{-\frac{1}{2}(\frac{1}{2}(N-3)+2\omega)} \tag{32}$$

$$\times I_{\frac{1}{2}(N-3)+2\omega}(\sqrt{u_1+u_2}),$$

where u_1 and u_2 are the two non-zero roots of (29).

(iv) *When $t > 2$, the joint distribution of the a_{ij} can be written by means of expression* (14) *as a multiple integral. The explicit form of the distribution has not yet been obtained.*

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[6] One method of demonstrating this fact is to apply (8) to centrally distributed variates and compare the Wishart distribution of the transformed variates with the Wishart distribution of the original variates.

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STATISTICAL TESTS BASED ON PERMUTATIONS OF THE OBSERVATIONS

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1. Introduction. One of the problems of statistical inference is to devise exact tests of significance when the form of the underlying probability distribution is unknown. The idea of a general method of dealing with this problem originated with R. A. Fisher [13, 14]. The essential feature of this method is that a certain set of permutations of the observations is considered, having the property that each permutation is equally likely under the hypothesis to be tested. Thus, an exact test on the level of significance α can be constructed by choosing a proportion α of the permutations as critical region. In an interesting paper H. Scheffé [2] has shown that for a general class of problems this is the only possible method of constructing exact tests of significance.

Tests based on permutations of the observations have been proposed and studied by R. A. Fisher, E. J. G. Pitman, B. L. Welch, the present authors, and others. Pitman and Welch derived the first few moments of the statistics used in their test procedures. However, it is desirable to derive at least the limiting distributions of these statistics and make it practicable to carry out tests of significance when the sample is large. Such a large sample distribution was derived for a statistic considered elsewhere [1] by the present authors.

In this paper a general theorem on the limiting distribution of linear forms in the universe of permutations of the observations is derived. As an application of this theorem, the limiting distributions of the rank correlation coefficient and that of several statistics considered by Pitman and Welch, are obtained. In the last section the limiting distribution of Hotelling's generalized T in the universe of permutations of the observations is derived.

2. A theorem on linear forms. Let $H_N = (h_1, h_2 \ldots, h_N)$ $(N = 1, 2, \ldots,$ ad inf.) be sequences of real numbers and let

$$\mu_r(H_N) = N^{-1} \sum_{\alpha=1}^{N} \left((h_\alpha - N^{-1} \sum_{\beta=1}^{N} h_\beta \right)^r$$

for all integral values of r. We define the following symbols in the customary manner: For any function $f(N)$ and any positive function $\varphi(N)$ let $f(N) = O(\varphi(N))$ mean that $| f(N)/\varphi(N) |$ is bounded from above for all N and let

$$f(N) = \Omega(\varphi(N))$$

mean that

$$f(N) = O(\varphi(N))$$

and that

$$\lim_N \inf | f(N)/\varphi(N) | > 0.$$

Also let

$$f(N) = o(\varphi(N))$$

mean that

$$\lim_{N\to\infty} f(N)/\varphi(N) = 0.$$

Let $[\rho]$ denote the largest integer $\leq \rho$.

We shall say that the sequences $H_N(N = 1, 2, \cdots, \text{ad inf.})$ satisfy the condition W if, for all integral $r > 2$,

$$\frac{\mu_r(H_N)}{[\mu_2(H_N)]^{r/2}} = O(1). \tag{2.1}$$

For any value of N let

$$X = (x_1, x_2, \cdots, x_N)$$

be a chance variable whose domain of definition is made up of the $N!$ permutations of the elements of the sequence $A_N = (a_1, a_2, \cdots, a_N)$. (If two of the $a_i (i = 1, 2, \cdots, N)$ are identical we assume that some distinguishing index is attached to each so that they can then be regarded as distinct and so that there still are $N!$ permutations of the elements $a_1, \cdots, a_N$). Let each permutation of A_N have the same probability $(N!)^{-1}$. Let $E(Y)$ and $\sigma^2(Y)$ denote, respectively, the expectation and variance of any chance variable Y.

We now prove the following:

THEOREM. *Let the sequences $A_N = (a_1, a_2, \cdots, a_N)$ and $D_N = (d_1, d_2, \cdots, d_N)$ $(N = 1, 2, \cdots, \text{ad inf.})$ satisfy the condition W. Let the chance variable L_N be defined as*

$$L_N = \sum_{i=1}^{N} d_i x_i .$$

Then as $N \to \infty$, the probability of the inequality

$$L_N - E(L_N) < t\,\sigma\,(L_N)$$

for any real t, approaches

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{1}{2}x^2}\, dx.$$

For convenience the proof will be divided into several lemmas.

Since

$$L_N^* = \frac{L_N - E(L_N)}{\sigma(L_N)}$$

remains invariant if a constant is added to all the elements of D_N or of A_N, or if the elements of either of the latter are multiplied by any constant other than zero, we may, in the formation of L_N^*, replace A_N and D_N by the sequences A_N'

and D'_N, respectively, whose ith elements a'_i and $d'_i (i = 1, 2, \cdots, N)$ are, respectively

$$a'_i = [\mu_2(A_n)]^{-\frac{1}{2}} \left(a_i - N^{-1} \sum_{j=1}^{N} a_j \right) \tag{2.2}$$

and

$$d'_i = [\mu_2(D_N)]^{-\frac{1}{2}} \left(d_i - N^{-1} \sum_{j=1}^{N} d_j \right). \tag{2.3}$$

The sequences A'_N and D'_N satisfy the condition W. Furthermore,

$$\mu_1(A'_N) \equiv \mu_1(D'_N) \equiv 0 \tag{2.4}$$

and

$$\mu_2(A'_N) \equiv \mu_2(D'_N) \equiv 1. \tag{2.5}$$

LEMMA 1.

$$\sum_{\alpha_1 < \alpha_2 < \cdots < \alpha_k \leq N} \cdots \sum a'_{\alpha_1} a'_{\alpha_2} \cdots a'_{\alpha_k} = O(N^{[k/2]}) \tag{2.6}$$

$$\sum_{\alpha_1 < \alpha_2 < \cdots < \alpha_k \leq N} \cdots \sum d'_{\alpha_1} d'_{\alpha_2} \cdots d'_{\alpha_k} = O(N^{[k/2]}). \tag{2.7}$$

From (2.4), (2.5), and the fact that the A'_N and D'_N satisfy condition W, it follows that the A'_N and D'_N satisfy conditions *a*), *b*), and *c*) of the theorem on page 383 of [1]. Our lemma 1 is the same as lemma 1 of [1].

LEMMA 2. *Let*

$$V = (v_1, v_2, \cdots, v_N)$$

be the same permutation of the elements of A'_N that X is of the elements of A_N. Let $y = v_1 \cdots v_k z$ where $z = v^{i_2}_{(k+1)} \cdots v^{i_r}_{(k+r)}$, $i_j > 1$ $(j = 1, 2, \cdots, r)$, and $k, r, i_1, \cdots, i_r$ are fixed values independent of N.
Then

$$E(y) = O(N^{[k/2]-k}). \tag{2.8}$$

This is Lemma 2 of [1].

In a similar manner we obtain that

$$\sum_{\alpha_1, \alpha_2, \cdots, \alpha_{(k+r)}} \cdots \sum d'_{\alpha_1} \cdots d'_{\alpha_k} d'^{i_1}_{\alpha_{(k+1)}} \cdots d'^{i_r}_{\alpha_{(k+r)}} = O(N^{[k/2]-k}) \cdot O(N^{k+r}) = O(N^{[k/2]+r}). \tag{2.9}$$

The summation in the above formula is to be taken over all possible sets of $k + r$ distinct positive integers $\leq N$.

LEMMA 3. *Let $\alpha_1, \cdots, \alpha_{(k+r)}$ be $(k + r)$ distinct positive integers $\leq N$. Then*

$$E(v_1 v_2 \cdots v_k v^{i_1}_{(k+1)} \cdots v^{i_r}_{(k+r)}) = E(v_{\alpha_1} v_{\alpha_2} \cdots v_{\alpha_k} v^{i_1}_{\alpha_{(k+1)}} \cdots v^{i_r}_{\alpha_{(k+r)}}). \tag{2.10}$$

This follows from the fact that all permutations of A'_N have the same probability.

LEMMA 4. *Let*

$$L'_N = \sum_{i=1}^{N} d'_i v_i .$$

Then

$$E(L'^p_N) = O(N)^{[p/2]}). \tag{2.11}$$

PROOF: Expand L'^p_N and take the expected value of the individual terms. The contribution to $E(L'^p_N)$ of all the terms which are multiples of the type appearing in the right member of (2.10) with fixed $k, r, i_1, \cdots, i_r (k + i_1 + \cdots + i_r = p)$, is, by Lemmas 2 and 3

$$O(N^{[k/2]-k}) \cdot \sum_{\substack{\alpha_1,\cdots,\alpha_{(k+r)} \\ \text{all different}}} \cdots \sum d'_{\alpha_1} \cdots d'_{\alpha_k} d'^{i_1}_{\alpha_{(k+1)}} \cdots d'^{i_r}_{\alpha_{(k+r)}} = O(N^{[k/2]-k}) O(N^{[k/2]+r})$$

$$= O(N^{2[k/2]-k+r}).$$

Since $i_j > 1 (j = 1, \cdots, r)$, it follows from the fact that $k + i_1 + \cdots + i_r = p$ that $2r \leq p - k$ and that $2r = p - k$ only if $i_1 = \cdots = i_r = 2$. Now

$$2\left[\frac{k}{2}\right] - k + r \leq r \leq \frac{p-k}{2} \leq \frac{p}{2}. \tag{2.12}$$

Hence the maximum value of $2\left[\frac{k}{2}\right] - k + r$ is reached when $r = \left[\frac{p}{2}\right]$ and $k = 0$. This proves the lemma.

From the last remarks of the preceding paragraph we obtain

LEMMA 5.

$$E(L'^{2j}_N) - \frac{(2j)!}{j!2^j} \Big(\sum_{\substack{\alpha_1,\cdots,\alpha_j \\ \text{all different}}} \cdots \sum d'^2_{\alpha_1} \cdots d'^2_{\alpha_j}\Big) E(v_1^2 \cdots v_j^2) = o(N^j). \tag{2.13}$$

We now prove

LEMMA 6.

$$E(L'_N) = 0 \tag{2.14}$$

$$E(L'^2_N) = NE(v_1^2) + o(N) = N + o(N). \tag{2.15}$$

Equation (2.14) follows from (2.2). Consider the expectations of the various terms in the expansion of L'^2_N. The sum of all the terms of the type

$$d'_i d'_j E(v_i v_j)$$

is

$$\Big(\sum_{i \neq j} d'_i d'_j\Big) E(v_1 v_2) = O(N) O(N^{-1}) = O(1),$$

by Lemmas 1 and 2. The sum of all the terms of the type

$$d'^2_i E(v_i^2)$$

is

$$\left(\sum_{i=1}^{N} d_i'^2\right) E(v_1^2) = NE(v_1^2) = N,$$

by (2.2) and (2.3). This proves the lemma.

LEMMA 7.

$$E(v_1^2 \cdots v_j^2) = 1 + o(1) \tag{2.16}$$

$$\sum_{\substack{\alpha_1 \cdots \alpha_j \\ \text{all different}}} \cdots \sum d_{\alpha_1}'^2 \cdots d_{\alpha_j}'^2 = N^j + o(N^j). \tag{2.17}$$

From (2.2) and (2.3), and Lemma 3, it follows that it will be sufficient to prove (2.17), because (2.16) follows in the same manner. Consider the relation

$$N^j = \left(\sum_{i=1}^{N} d_i'^2\right)^j = \sum_{\substack{\alpha_1, \cdots, \alpha_j \\ \text{all different}}} \cdots \sum d_{\alpha_1}'^2 \cdots d_{\alpha_j}'^2 + \text{other terms.}$$

By (2.9) the sum of these other terms must be not larger than $O(N^{j-1})$. From this follows the lemma.

PROOF of the theorem: Since

$$L_N^* = \frac{L_N'}{\sigma(L_N')} = \frac{L_N - E(L_N)}{\sigma(L_N)},$$

it will be sufficient to show that the moments of L_N^* approach those of the normal distribution as $N \to \infty$. From (2.14), (2.15), and (2.11) we see that, when p is odd, the pth moment of L_N^* is $O(N^{-\frac{1}{2}})$ and hence approaches zero as $N \to \infty$. When p is even and $= 2s$ (say), it follows from Lemma 5 that

$$E(L_N'^{2s}) - \frac{(2s)!}{s!2^s} \left(\sum_{\substack{\alpha_1, \cdots, \alpha_s \\ \text{all different}}} \cdots \sum d_{\alpha_1}'^2 \cdots d_{\alpha_s}'^2\right) E(v_1^2 \cdots v_s^2) = o(N^s).$$

Hence from (2.16) and (2.17)

$$E(L_N'^{2s}) = \frac{(2s)!}{s!2^s} N^s + o(N^s). \tag{2.18}$$

From (2.18) and (2.15) we obtain that

$$\lim_{N \to \infty} E(L_N^{*2s}) = \frac{(2s)!}{s!2^s}.$$

This completes the proof of the theorem.

It will be noticed that nothing in the foregoing proof requires that, when $N < N'$, the sequences A_N and D_N be subsequences of $A_{N'}$ and $D_{N'}$. Indeed, the sequences were written as they were simply for typographic brevity. We have therefore

COROLLARY 1. *The theorem is valid for sequences*

$$A_N = (a_{N1}, \cdots, a_{NN})$$

$$D_N = (d_{N1}, \cdots, d_{NN})$$

$$(N = 1, 2, \cdots \text{ad inf.})$$

provided they fulfill condition W.

COROLLARY 2. *If the elements $a_i (i = 1, 2, \cdots \text{ad inf.})$ are all independent observations on the same chance variable, all of whose moments are finite and whose variance is positive, the sequences $A_N (N = 1, 2, \cdots, \text{ad inf.})$ will fulfill condition W with probability one.*

3. The rank correlation coefficient. For this well known statistic (see [3])

$$A_N \equiv D_N \equiv (1, 2, 3, \cdots, N).$$

The sequences A_N and D_N satisfy the condition W. For

$$\sum_{i=1}^{N} i^r = O(N^{r+1})$$

and hence, for $r \geq 3$

$$\mu_r(A_N) = \mu_r(D_N) = O(N^r).$$

Also

$$\mu_2(A_N) = \mu_2(D_N) = \Omega(N^2).$$

Hence the distribution of the rank correlation coefficient is asymptotically normal in the case of statistical independence. This result was first proved by Hotelling and Pabst [3].

4. Pitman's test for dependence between two variates. The distribution of the correlation coefficient in the population of permutations of the observations was used by Pitman [4] in a test for dependence between two variates which "involves no assumptions" about the distributions of these variates. In our notation, let

$$(a_i, d_i)(i = 1, 2, \cdots, N)$$

be N observations on the pair of variates A and D whose dependence it is desired to test. Then the value of the correlation coefficient is

$$N^{-1} \sum_{i=1}^{N} d_i' a_i' .$$

At the level β the observations are considered to be significant if the probability that $N^{-1} \mid L_N' \mid$ be equal to or greater than the absolute value of the actually observed correlation coefficient is $\leq \beta$.

In his paper ([4], page 227] Pitman points out that if the ratios of certain sample cumulants are "not too large," then, as $N \to \infty$, the first four moments of $N^{-\frac{1}{2}} L'_N$ will approach 0, 1, 0, and 3, respectively (the first moment is always zero). Our theorem and the relation (2.15) make clear that under proper circumstances all the moments will approach those of the normal distribution.

5. Pitman's procedure for testing the hypothesis that two samples are from the same population. For testing the hypothesis that two samples came from the same population Pitman [5] proposed the following procedure:
Let one sample be

$$a_1, a_2, \cdots, a_m$$

and the other

$$a_{m+1}, a_{m+2}, \cdots, a_{m+n}.$$

Write $m + n = N$, and construct the sequences A_N and A'_N as before defined. Let

$$d_i = 1 \qquad (i = 1, \cdots, m)$$

$$d_i = 0 \qquad (i = m + 1, \cdots, N)$$

and construct the sequences D_N and D'_N. Then the value of the statistic considered by Pitman is, except for a constant factor,

$$N^{-\frac{1}{2}} \left(\sum_{i=1}^{N} d'_i a'_i \right). \tag{5.1}$$

At the level β the observations are considered significant if the probability that $N^{-\frac{1}{2}} \mid L'_N \mid$ be equal to or greater than the observed absolute value of the expression (5.1) is $\leq \beta$.

Let $N \to \infty$, while $\frac{m}{n}$ is constant. Then the sequences D_N are seen to satisfy condition W. If then the sequences A_N satisfy condition W we may, for large N, employ the result of our theorem and expeditiously determine the critical value of Pitman's statistic.

6. Analysis of variance in randomized blocks. Welch [7] and Pitman [6] consider the following problem: Each of n different "varieties of a plant" is planted in one of the n cells which constitute a "block." It is desired to test, on the basis of results from m blocks, the null hypothesis that there is no difference among the varieties. In order to eliminate a possible bias caused by variations in fertility among the cells of a block, the varieties are assigned at random to the cells of a block. If the cells of the jth block are designated by $(j1), (j2), \cdots, (jn)$, a permutation of the integers $1, 2, \cdots, n$ is allocated to the jth block by a chance process, each permutation having the same probability $(n!)^{-1}$.

Let x_{ijk} be the yield of the ith variety in the kth cell of the jth block to which it was assigned by the randomization process. It is assumed that

$$x_{ijk} = y_{jk} + \delta_i + \epsilon_{jk},$$

where y_{jk} is the "effect" of the kth cell in the jth block, δ_i is the "effect" of the ith variety, and ϵ_{jk} are chance variables about whose distribution we assume nothing. The null hypothesis states that

$$\delta_1 = \delta_2 = \cdots = \delta_n = 0.$$

Let a_{jk} be the yield in the kth cell of the jth block and x_{ij} the yield of the ith variety in the jth block. If the null hypothesis is true then, because of the randomization within each block described above, the conditional probability that, given the set $\{a_{jk}\}(k = 1, 2, \cdots, n)$, the sequence $x_{1j}, x_{2j} \cdots, x_{nj}$, be any given permutation of the elements of $\{a_{jk}\}$ is $(n!)^{-1}$. Permuting in all the blocks simultaneously we have that, under the null hypothesis, given the set of mn values $\{a_{jk}\}$ $(j = 1, 2, \cdots m; k = 1, 2, \cdots, n)$, the conditional probability of any of the permutations is the same, $(n!)^{-m}$. This permits an exact test of the null hypothesis.

The classical analysis of variance statistic that would be employed in the conventional two-way classification with independent normally distributed observations is

$$F = \frac{(m-1)m \sum (x_{i\cdot} - x)^2}{\sum\sum (x_{ij} - x_{i\cdot} - x_{\cdot j} + x)^2}$$

where

$$x_{i\cdot} = m^{-1} \sum_j x_{ij} \qquad x = (mn)^{-1} \sum\sum x_{ij}.$$
$$x_{\cdot j} = n^{-1} \sum_i x_{ij}$$

The statistic W used by Welch and Pitman is

$$W = F(m - 1 + F)^{-1}.$$

Since W is a monotonic function of F and the critical regions are the upper tails, the two tests are equivalent. The distribution of F or W is to be determined in the same manner as that of the other statistics discussed in this paper, i.e., over the equally probable permutations of the values actually observed. The critical region is, as usual, the upper tail.

Since x_{ij} takes any of the values $a_{j1}, \cdots, a_{jn}$ with probability $1/n$, we have

$$E(x_{ij}) = n^{-1} \sum_k a_{jk} = a_j \quad \text{(say)}. \tag{6.1}$$

$$\sigma^2(x_{ij}) = n^{-1} \sum_k (a_{jk} - a_j)^2 = b_j \quad \text{(say)}. \tag{6.2}$$

$$\begin{aligned} \sigma(x_{i_1 j} x_{i_2 j}) &= [n(n-1)]^{-1} \sum_{k_1 \neq k_2} a_{jk_1} a_{jk_2} - a_j^2 \\ &= [n(n-1)]^{-1} [(\sum_k a_{jk})^2 - \sum_k a_{jk}^2] - a_j^2 \\ &= [n^2 a_j^2 - \sum_k a_{jk}^2] [n(n-1)]^{-1} - a_j^2 \\ &= (n-1)^{-1} [a_j^2 - n^{-1} \sum_k a_{jk}^2] = -(n-1)^{-1} b_j. \end{aligned} \tag{6.3}$$

Hence

$$E(x_{i\cdot}) = m^{-1} \sum a_j . \tag{6.4}$$

$$\sigma^2(x_{i\cdot}) = m^{-2} \sum b_j = b \quad \text{(say)}. \tag{6.5}$$

$$\sigma(x_{i_1\cdot}, x_{i_2\cdot}) = -[m^2(n-1)]^{-1} \sum b_j = c \quad \text{(say)}. \tag{6.6}$$

$$i_1 \neq i_2$$

Let

$$x^*_{ij} = \sum_v \lambda_{iv} x_{vj} \qquad (i, v = 1, \cdots, n)$$

where $|| \lambda_{iv} ||$ is an orthogonal matrix and

$$\lambda_{n1} = \lambda_{n2} = \cdots = \lambda_{nn} = n^{-\frac{1}{2}}.$$

Then it follows that

$$E(x^*_{i\cdot}) = 0$$

$$\sigma^2(x^*_{i\cdot}) = b - c \qquad (i = 1, 2, \cdots, n-1) \tag{6.7}$$

$$\sigma(x^*_{i_1\cdot}, x^*_{i_2\cdot}) = 0 \qquad (i_1 \neq i_2 ; i_1, i_2 = 1, \cdots, n-1).$$

Furthermore, we have

$$\sum_{i=1}^{n-1} x^{*2}_{i\cdot} = \sum_{i=1}^{n} (x_{i\cdot} - x)^2. \tag{6.8}$$

Applying the well known identity

$$\Sigma\Sigma(x_{ij} - x_{i\cdot} - x_{\cdot j} + x)^2 = \Sigma\Sigma(x_{ij} - x_{\cdot j})^2 - m\Sigma(x_{i\cdot} - x)^2$$

to the definitions of F and W we obtain

$$W = \frac{m \sum_i (x_{i\cdot} - x)^2}{\sum_i \sum_j (x_{ij} - x_{\cdot j})^2}. \tag{6.9}$$

The denominator of the right member of (6.9) is invariant under permutations *within* each block and equals

$$\sum_j \sum_k (a_{jk} - a_j)^2 = (n-1)m^2(b-c).$$

Hence

$$\begin{aligned} W &= [m(n-1)(b-c)]^{-1} \sum_{i=1}^{n} (x_{i\cdot} - x)^2 \\ &= [m(n-1)(b-c)]^{-1} \sum_{i=1}^{n-1} x^{*2}_{i\cdot}. \end{aligned} \tag{6.10}$$

If the joint distribution of the $x^*_{i\cdot} (i = 1, 2, \cdots, n-1)$ over the set of admissible permutations approaches a normal distribution with non-singular correlation

matrix as m, the number of blocks, becomes large, it follows from (6.7) and (6.10) that the distribution of $m\ (n-1)\ W$ approaches the x^2 distribution with $n-1$ degrees of freedom. Hence it remains to indicate conditions on the set $\{a_{jk}\}$ which would make the distribution of the $x_{i\cdot}^*$ approach normality. Each $x_{i\cdot}^*$ is the mean of independent variables, so these conditions need not be very restrictive.

According to Cramér [8], Theorem 21a, page 113, if the variances and covariances fulfill certain requirements (the limiting correlation matrix should also be non-singular) and if a generalized Lindeberg condition holds, normality in the limit will follow. Somewhat more restrictive conditions which are simpler to state and which will be satisfied in most statistical applications are that $o < c' < b_j < c''$ for all j, where c' and c'' are positive constants. Since the variance of x_{ij}^* is $(n-1)^{-1}nb_j$, it can be seen that the above inequalities imply the fulfillment of the conditions of the Laplace-Liapounoff theorem (see, for example, Uspensky [9], page 318). By [6.7] the correlation matrix is non-singular.

7. Hotelling's generalized T for permutation of the observations. In this section we shall restrict ourselves to bivariate populations, the extension to more than two variables being straightforward. Let $(u_{11}, u_{21}), \cdots, (u_{1m}, u_{2m})$ be m pairs of observations on the chance variables U_1, U_2, and $(u_{1(m+1)}, u_{2(m+1)}), \cdots, (u_{1N}, u_{2N})$, be n pairs of observations on the chance variables U_1', U_2', where $m + n = N$. If each of the pairs U_1, U_2, and U_1', U_2' is jointly normally distributed with the same convariance matrix, the Hotelling generalized T for testing the null hypothesis that

$$E(U_1) = E(U_1') \tag{7.1}$$

and

$$E(U_2) = E(U_2'), \tag{7.2}$$

is given (Hotelling [10]) by

$$T^2 = N^{-1}(mn) \sum_{j=1}^{2} \sum_{i=1}^{2} q_{ij}(\bar{u}_i - \bar{u}_i')(\bar{u}_j - \bar{u}_j')$$

where

$$m\bar{u}_i = \sum_{l=1}^{m} u_{il} \quad n\bar{u}_i' = \sum_{l=m+1}^{N} u_{il}$$

and the matrix $\| q_{ij} \|$ is the inverse of the matrix $\| b_{ij} \|$ with b_{ij} given by

$$(N-2)b_{ij} = \sum_{l=1}^{m} (u_{il} - \bar{u}_i)(u_{jl} - \bar{u}_j) + \sum_{l=m+1}^{N} (u_{il} - \bar{u}_i')(u_{jl} - \bar{u}_j').$$

In Hotelling's procedure the b_{ij} are sample estimates of the population covariances whose distribution is independent of that of the sample means. A constant multiple of the statistic T^2 has the analysis of variance distribution under the null hypothesis. If the population covariances were known and used in place of the b_{ij}, T^2 would have the χ^2 distribution with two degrees of freedom.

Let us now apply the generalized T over the permutations of the actually observed values, as was done with other statistics in previous sections. If we do this literally we will find that the b_{ij} are no longer independent of the sample means. To avoid this complication we shall use a slightly different statistic T' which, as will be shown later, is a monotonic function of T, so that the test based on T' is identical with that based on T. The statistic T' is defined as follows: Let

$$\bar{U}_i = N^{-1} \sum_{k=1}^{N} u_{ik}$$

$$c'_{ij} = N[(N-1)mn]^{-1} \sum_{k=1}^{N} (u_{ik} - \bar{U}_i)(u_{jk} - \bar{U}_j) \qquad (i, j, = 1, 2)$$

and

$$\| q'_{ij} \| = \| c'_{ij} \|^{-1}.$$

Then

$$T'^2 = \sum_{i=1}^{2} \sum_{j=1}^{2} q'_{ij}(\bar{u}_i - \bar{u}'_i)(\bar{u}_j - \bar{u}'_j). \tag{7.3}$$

The expression T'^2 is much simpler than T^2, since the coefficients q'_{ij} are constants in the population of permutations of the observations. We shall show that T'^2 is a monotonic function of T^2. Let

$$Q_{ij} = \sum_{k=1}^{m} (u_{ik} - \bar{u}_i)(u_{jk} - \bar{u}_j) + \sum_{k=m+1}^{N} (u_{ik} - \bar{u}'_i)(u_{jk} - \bar{u}'_j)$$

$$Q'_{ij} = \sum_{k=1}^{N} (u_{ik} - \bar{U}_i)(u_{jk} - \bar{U}_j)$$

$$\| Q^{ij} \| = \| Q_{ij} \|^{-1}$$

$$\| Q'^{ij} \| = \| Q'_{ij} \|^{-1}.$$

Then the expressions

$$T_1^2 = \sum_{i=1}^{2} \sum_{j=1}^{2} Q^{ij}(\bar{u}_i - \bar{u}'_i)(\bar{u}_j - \bar{u}'_j) \tag{7.4}$$

and

$$T_2^2 = \sum_{i=1}^{2} \sum_{j=1}^{2} Q'^{ij}(\bar{u}_i - \bar{u}'_i)(\bar{u}_j - \bar{u}'_j), \tag{7.5}$$

are constant multiples of T^2 and T'^2, respectively. Hence it is sufficient to show that T_2^2 is a monotonic function of T_1^2. We have

$$Q'_{ij} = Q_{ij} + m(\bar{u}_i - \bar{U}_i)(\bar{u}_j - \bar{U}_j) + n(\bar{u}'_i - \bar{U}_i)(\bar{u}'_j - \bar{U}_j). \tag{7.6}$$

Furthermore, we have

$$\bar{u}_i - \bar{U}_i = \bar{u}_i - \frac{m\bar{u}_i + n\bar{u}'_i}{m+n} = \frac{n(\bar{u}_i - \bar{u}'_i)}{m+n}. \tag{7.7}$$

Similarly

$$\bar{u}_i' - \overline{U}_i = \bar{u}_i' - \frac{m\bar{u}_i + n\bar{u}_i'}{m+n} = \frac{m(\bar{u}_i' - \bar{u}_i)}{m+n}. \tag{7.8}$$

From (7.6), (7.7) and (7.8) it follows that

$$\begin{aligned} Q_{ij}' &= Q_{ij} + \frac{mn^2}{(m+n)^2}(\bar{u}_i - \bar{u}_i')(\bar{u}_j - \bar{u}_j') + \frac{nm^2}{(m+n)^2}(\bar{u}_i - \bar{u}_i')(\bar{u}_j - \bar{u}_j') \\ &= Q_{ij} + \frac{mn}{m+n}(\bar{u}_i - \bar{u}_i')(\bar{u}_j - \bar{u}_j'). \end{aligned} \tag{7.9}$$

Denote $\frac{mn}{m+n}$ by λ and $\bar{u}_i - \bar{u}_i'$ by h_i. Then we have

$$Q_{ij}' = Q_{ij} + \lambda h_i h_j. \tag{7.10}$$

Denote the cofactor of Q_{ij} in $\| Q_{ij} \|$ by R_{ij} and the cofactor of Q_{ij}' in $\| Q_{ij}' \|$ by R_{ij}'. Then

$$\frac{|Q_{ij}|}{|Q_{ij}'|} = \frac{|Q_{ij}|}{|Q_{ij} + \lambda h_i h_j|} = \frac{|Q_{ij}|}{|Q_{ij}| + \lambda\Sigma\Sigma R_{ij} h_i h_j} = \frac{1}{1 + \lambda T_1^2}. \tag{7.11}$$

Furthermore, we have

$$\frac{|Q_{ij}|}{|Q_{ij}'|} = \frac{|Q_{ij}' - \lambda h_i h_j|}{|Q_{ij}'|} = \frac{|Q_{ij}'| - \lambda\Sigma\Sigma R_{ij}' h_i h_j}{|Q_{ij}'|} = 1 - \lambda T_2^2. \tag{7.12}$$

From (7.11) and (7.12) it follows that T_2^2 is a monotonic function of T_1^2. Hence also T'^2 is a monotonic function of T^2 and, therefore, we do not change our test procedure by using T'^2 instead of T^2.

Let the sequence of pairs

$$(x_{11}, x_{21}), \cdots, (x_{1N}, x_{2N})$$

be a permutation of the actually observed pairs

$$(u_{11}, u_{21}), \cdots, (u_{1N}, u_{2N})$$

where to each permutation is ascribed the same probability $(N!)^{-1}$. Then one obtains for $i = 1, 2$,

$$E(\bar{x}_i - \bar{x}_i') = 0 \tag{7.13}$$

$$\sigma^2(\bar{x}_i - \bar{x}_i') = N[(N-1)mn]^{-1}\sum_{j=1}^{N}(u_{ij} - \overline{U}_i)^2 = c_{ii}' \tag{7.14}$$

$$E(\bar{x}_1 - \bar{x}_1')(\bar{x}_2 - \bar{x}_2') = N[(N-1)mn]^{-1}\sum_{j=1}^{N}(u_{1j} - \overline{U}_1)(u_{2j} - \overline{U}_2) = c_{12}'. \tag{7.15}$$

Hence $\| c_{ij}' \|$ is the covariance matrix of the variates

$$(\bar{x}_1 - \bar{x}_1') \quad \text{and} \quad (\bar{x}_2 - \bar{x}_2').$$

Now we shall show that the limiting distribution of T'^2, as $N \to \infty$, is the χ^2 distribution with 2 degrees of freedom, provided that the observation u_{ik} $(i = 1, 2; k = 1, \cdots, N)$ satisfy some slight restrictions. Since $|| q'_{ij} ||$ is the inverse of the covariance matrix $|| c'_{ij} ||$ our statement about the limiting distribution of T'^2 is obviously proved if we can show that $\bar{x}_1 - \bar{x}'$ and $\bar{x}_2 - \bar{x}'_2$ have a joint normal distribution in the limit.

Let $N \to \infty$ while m/n remains constant. Let the sequences A_N and D_N of Section II be defined as follows:

There are two sequences A_N, denoted respectively by A_{1N} and A_{2N}, such that

$$a_{ij} = u_{ij} \qquad (i = 1, 2; j = 1, \cdots, N).$$

Also

$$d_j = \frac{1}{m} \qquad (j = 1, \cdots, m)$$

$$d_j = -\frac{1}{n} \qquad (j = m + 1, \cdots, N).$$

Then the sequences D_N satisfy the condition W. If also the sequences A_{iN} satisfy the condition W, the distribution of $\bar{x}_i - \bar{x}'_i$ approaches the normal distribution as N increases, by the theorem of Section 2. If the joint distribution of $\bar{x}_1 - \bar{x}'_i$ and $\bar{x}_2 - \bar{x}'_2$ approaches a normal distribution with non-singular correlation matrix, the distribution of T'^2 approaches that of χ^2 with two degrees of freedom.

The correlation matrix of $(\bar{x}_1 - \bar{x}'_1)$ and $(\bar{x}_2 - \bar{x}'_2)$ will be of rank two in the limit if the correlation coefficient between $(\bar{x}_1 - \bar{x}'_1)$ and $(\bar{x}_2 - \bar{x}'_2)$ approaches a limit ρ, where $|\rho| < 1$. By (7.14) and (7.15) this is equivalent to saying that the absolute value of the angle between the vectors A'_{1N} and A'_{2N} is eventually greater than a positive lower bound. We shall show that, if the correlation coefficient approaches, as $N \to \infty$, a limit ρ whose absolute value is less than one, and if A_{1N} and A_{2N} satisfy the condition W, then $(\bar{x}_1 - \bar{x}'_1)$ and $(\bar{x}_2 - \bar{x}'_2)$ are *jointly* normally distributed in the limit.

Let δ_1 and δ_2 be any two real numbers not both zero. Then the sequence

$$A^*_N = (a^*_1, \cdots, a^*_N)$$

where

$$a^*_j = \delta_1 a_{1j} + \delta_2 a_{2j}$$

will be shown to satisfy the condition W. If either δ_1 or δ_2 is zero this is trivial; assume therefore that neither is zero. Without loss of generality we may assume that $\sum_{j=1}^{N} a_{ij} = 0$, for if this were not so we could replace the original a_{ij} by $a'_{ij} = a_{ij} - N^{-1} \sum_j a_{ij}$ as was done in Section 2. Let ρ' be such that $1 > \rho' > |\rho|$.

For N sufficiently large we have

$$\mu_2(A_N^*) \geq N^{-1}(\delta_1^2 \sum_j a_{1j}^2 - 2 \mid \delta_1 \delta_2 \mid \sum a_{1j} a_{2j} \mid + \delta_2^2 \sum_j a_{2j}^2)$$

$$\geq N^{-1}(\delta_1^2 \sum_j a_{1j}^2 - 2\rho' \mid \delta_1 \delta_2 \mid \sqrt{(\sum_j a_{1j}^2)(\sum_j a_{2j}^2)} + \delta_2^2 \sum_j a_{2j}^2)$$

$$= N^{-1}[(\mid \delta_1 \mid \sqrt{\sum_j a_{1j}^2} - \mid \delta_2 \mid \sqrt{\sum_j a_{2j}^2})^2$$

$$+ 2(1 - \rho') \mid \delta_1 \delta_2 \mid \sqrt{(\sum_j a_{1j}^2)(\sum_j a_{2j}^2)}]$$

and

$$\mu_2(A_N^*) \leq 2(\delta_1^2 \mu_2(A_{1N}) + \delta_2^2 \mu_2(A_{2N})).$$

Hence

$$\mu_2(A_N^*) = \Omega[\max \{\mu_2(A_{1N}), \mu_2(A_{2N})\}]. \tag{7.16}$$

Also $\mu_r(A_N^*)$ is a sum of constant multiples of terms of the type

$$N^{-1} \sum_j a_{1j}^i a_{2j}^{r-i}.$$

By Schwarz' inequality

$$N^{-1} \sum_j a_{1j}^i a_{2j}^{r-i} \leq N^{-1}(\sum_j a_{1j}^{2i})^{\frac{1}{2}}(\sum_j a_{2j}^{2(r-i)})^{\frac{1}{2}} = (\mu_{2i}(A_{1N})\mu_{2(r-i)}(A_{2N}))^{\frac{1}{2}}. \tag{7.17}$$

The required result follows from (7.16) and (7.17).

Since the sequences A_N^* satisfy the condition W, the limiting distribution of

$$\delta_1(\bar{x}_1 - \bar{x}_1') + \delta_2(\bar{x}_2 - \bar{x}_2'),$$

for any pair δ_1, δ_2 not both zero, is normal. From this and a theorem of Cramér and Wold ([11] Theorem 1; see also [8], Theorem 31) it follows that if the joint distribution of $(\bar{x}_1 - \bar{x}_1')$ and $(\bar{x}_2 - \bar{x}_2')$ approaches a limit, this limit must be the normal distribution. From a theorem of Radon ([12]; see also Cramér [8], page 101) it follows that if the joint distribution of $(\bar{x}_1 - \bar{x}_1')$ and $(\bar{x}_2 - \bar{x}_2')$ does not approach a limit as $N \to \infty$ it is possible to find two subsequences of the sequence $(1, 2, \cdots, N, \cdots$ ad inf.) for each of which the joint distribution approaches a different limit. This contradicts the previous result. Hence the limit exists and is the normal distribution. This proves our statement that the limiting distribution of T'^2 is the χ^2 distribution with two degrees of freedom.

The statistic T'^2 seems to be appropriate for testing the null hypothesis that two bivariate distributions Π_1 and Π_2 are identical if the alternatives are restricted to the case where Π_2 differs from Π_1 only in the mean values, i.e., the distribution Π_2 can be obtained from Π_1 by a translation. This is no restriction as compared with Hotelling's T-test since also the T-test is based on the assumption that the two normal populations differ at most in their mean values, i.e., the covariance matrices in the two populations are assumed to be equal.

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ERROR CONTROL IN MATRIX CALCULATION

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1. Introduction. The solutions of large sets of simultaneous equations and the inversion of matrices are often complicated by the fact that errors, such as those introduced by rounding, become magnified in the course of the calculations to such an extent that the results are useless. In this paper we shall show that if the norm of the matrix $A - I$ is less than 0.35, operations involving the inversion A or the multiplication by A^{-1} will be in a state of error control for "Doolittle" methods of calculation. Thus such calculations may be carried through with assurance that the errors in the results will be limited to two or three significant figures. We also point out that as soon as an approximation to A^{-1} is available, most problems may be restated to bring them within the requirements for error control. Therefore the solution can be immediately completed to the desired degree of accuracy in one step instead of requiring multiple steps as do the iterative methods.

2. The inversion of special matrices. Consider the problem of inverting the matrix $(I + F)$ where I is the identity matrix and $(I + F)$ is a non-singular square matrix. Let

$$G = (I + F)^{-1}. \tag{2.1}$$

Then

$$(I + F)G = I \tag{2.2}$$

or

$$G = I - FG. \tag{2.3}$$

In ordinary algebra this would not be a practical formula for the calculation of G. However in matrix algebra the situation may be different. Examine the expanded form of G:

$$g_{ij} = \delta_{ij} - \Sigma f_{ik} g_{kj}. \tag{2.4}$$

The summation is over all values of k from 1 to n. Next examine the affect of imposing certain restrictions on F. For example, let $f_{ij} = 0$ if $j \geq i$. This is equivalent to making the summation in (2.4) over the range 1 to $i - 1$. The first row of (2.4) then becomes

$$g_{ij} = \delta_{ij}$$

and no g's appear on the right. For the second row

$$g_{2j} = \delta_{2j} - f_{21} g_{1j}.$$

The only g's on the right are those on the first row which have already been calculated. For the third row

$$g_{3j} = \delta_{3j} - f_{31}g_{1j} - f_{32}g_{2j} .$$

The only g's on the right are in the first and second rows and have already been calculated. Similarly for the fourth and later rows.

Thus it is seen that if F is a "pre-diagonal" matrix, (2.3) is a very simple and practical formula for the numerical calculation of the inverse of $(I + F)$. Also if F is a post-diagonal matrix, (2.3) may be used by working up from the bottom row.

Similarly, if a matrix H is to be multiplied by the inverse of $(I + F)$, let

$$G = (I + F)^{-1}H \tag{2.5}$$

and the working equation becomes

$$G = H - FG. \tag{2.6}$$

The inversion of a diagonal matrix is accomplished by inverting each of its diagonal elements. That is if

$$F = || \delta_{ij}s_{ii} || \tag{2.7}$$

then

$$F^{-1} = || \delta_{ij}s_{ii}^{-1} ||. \tag{2.8}$$

3. The inversion of general matrices. The general inversion problem will be solved if a general matrix can be factored into matrices of the special types treated in the last section. For the moment assume that such a factorization is possible and let the factors of the general matrix, A, be

$$A = (R_1 + I)S_1(I + T_1) \tag{3.1}$$

where R_1 is a prediagonal matrix, S_1 a diagonal matrix, and T_1 a postdiagonal matrix. Then

$$A = S_1 + R_1S_1 + S_1T_1 + R_1S_1T_1 . \tag{3.2}$$

A slight change in form now appears desirable so let

$$\begin{aligned} A &= (RS^{-1} + I)S(I + S^{-1}T) \\ &= R + S + T + RS^{-1}SS^{-1}T \end{aligned} \tag{3.3}$$

$$= R + S + T + RS^{-1}T. \tag{3.4}$$

For convenience let

$$B = R + S + T \tag{3.5}$$

and remember that R, S, and T have no non-zero elements in common. Therefore the non-zero elements of R, S, and T are equal to the corresponding elements of B. Rearranging (3.4) gives

$$B = A - RS^{-1}T \tag{3.6}$$

and the elements of B are determined by

$$b_{ij} = a_{ij} - \Sigma \frac{r_{ik} t_{kj}}{s_{kk}} . \tag{3.7}$$

Since $r_{ik} = 0$ for $k \geq i$, there is no point in making the summation beyond $k = i - 1$. Also since $t_{kj} = 0$ for $k \geq j$, there is no point in making the summation beyond $j - 1$. Therefore the summation in (3.7) is to be considered to be over the range 1 to the smaller of $i - 1$ and $j - 1$. The r's, s's and t's on the right of (3.7) can now be replaced by the corresponding b's:

$$b_{ij} = a_{ij} - \Sigma \frac{b_{ik} b_{kj}}{b_{kk}} . \tag{3.8}$$

Since the first row (column) of b's equal the first row (column) of a's, the second row (column) of b's is a function of only those b's in the first row and the first column, etc., any calculation routine which works down from the top and from the left to the right will lead to a ready determination of all the b's by (3.8).

Thus we see that the assumed factorization (3.3) of A is always possible (unless some of the diagonal elements, b_{kk}, of B are zero) and moreover the elements of the factors are readily calculated by the simple equations, (3.8).

Therefore, to invert the general non-singular square matrix A, calculate the elements of an intermediate matrix $B = R + S + T$ by equations (3.5) and (3.7). Then from (3.3) we have

$$A^{-1} = (I + S^{-1}T)^{-1} S^{-1}(I + RS^{-1})^{-1} \tag{3.9}$$

which can be readily calculated by the methods of (2.3) and (2.6).

4. The Doolittle method. The Doolittle method of matrix calculation can now be expressed in terms of the matrices R, S, and T studied above. To illustrate we shall use the set of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_{11}y_1 + c_{12}y_2 + c_{13}y_3 = d_1 , \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= c_{21}y_1 + c_{22}y_2 + c_{23}y_3 = d_2 , \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= c_{31}y_1 + c_{32}y_2 + c_{33}y_3 = d_3 . \end{aligned} \tag{4.1}$$

This set of equations will be represented in the form of a three element matrix,

$$\| AX : CY : D \| . \tag{4.1.a}$$

The essential feature of the Doolittle method of solution is that we replace (4.1) by an equivalent set of equations for which the prediagonal coefficients of the X's are all zero and the diagonal coefficients are all unity. Therefore consider the set formed as follows:

$$(4.2)\quad \| A_3X : C_3Y : D_3 \| = S^{-1}\{\| AX : CY : D \| - R \| A_3Y : C_3Y : D_3 \|\}.$$

Then

$$(4.3)\qquad A_3 = S^{-1}\{A - RA_3\}$$

or

$$(4.4)\qquad (S + R)A_3 = A$$

or

$$(4.5)\qquad \begin{aligned} A_3 &= (S + R)^{-1}(RS^{-1} + I)S(I + S^{-1}T) && \text{by (3.3)} \\ &= (I + S^{-1}T). \end{aligned}$$

Since $S^{-1}T$ is a postdiagonal matrix, $\| A_3X : C_3Y : D_3 \|$ are the required intermediate equations for a Doolittle type of solution.

The final solution is now easily obtained. Consider

$$(4.6)\quad \| A_4X : C_4Y : D_4 \| = \| A_3X : C_3Y : D_3 \| - (S^{-1}T) \| A_4X : C_4Y : D_4 \|.$$

We have

$$(4.7)\qquad A_4 = A_3 - (S^{-1}T)A_4$$

or

$$(4.8)\qquad \begin{aligned} (I + S^{-1}T)A_4 &= A_3 \\ &= I + S^{-1}T && \text{by (4.5).} \end{aligned}$$

Therefore A_4 is in fact the identity matrix and (4.6) can be rewritten

$$(4.9)\quad \| X : C_4Y : D_4 \| = \| A_3X : C_3Y : D_3 \| - (S^{-1}T) \| X : C_4Y : D_4 \|$$

5. The non-symmetric case. In actual practice, the work has to be so arranged that the elements of the matrices R, S, and $(S^{-1}T)$ are set out so as to be readily available for use as multipliers in forming the intermediate and final sets of equations. Table I gives such a practical layout for the non-symmetric case.

The elements of $(S^{-1}T)$ are set out as the postdiagonal elements of A_3 so that they do not need further attention. To determine the elements of R and S, we form a set of pre-intermediate equations:

$$(5.1)\quad \| BX : \cdots : \cdots \| = \| AX : \cdots : \cdots \| - R \| A_3X - X : \cdots : \cdots \|.$$

TABLE I

Layout of Doolittle solution for the non-symmetric case. Coefficients not used further are indicated by ⋯ .

$$
\begin{array}{rrrrrrrrrrrl}
a_{11}x_1 & + a_{12}x_2 & + a_{13}x_3 & = c_{11}y_1 & + c_{12}y_2 & + c_{13}y_3 & = d_1 \\
a_{21}x_1 & + a_{22}x_2 & + a_{23}x_3 & = c_{21}y_1 & + c_{22}y_2 & + c_{23}y_3 & = d_2 \\
a_{31}x_1 & + a_{32}x_2 & + a_{33}x_3 & = c_{31}y_1 & + c_{32}y_2 & + c_{33}y_3 & = d_3
\end{array}
$$

$$
\begin{array}{rrrrrrrrrrrl}
s_1x_1 & + \cdots & & \cdots & = \cdots & & \cdots & & \cdots & = d_{2:1} & = d_1 \\
x_1 & + a_{3:12}x_2 & + a_{3:13}x_3 & = c_{3:11}y_1 & + c_{3:12}y_2 & + c_{3:13}y_3 & = d_{3:1} & = d_1/s_1 \\
r_{21}x_1 & + s_2x_2 & + \cdots & = \cdots & & \cdots & \cdots & = d_{2:2} & = d_2 - r_{21}[d_{3:1} - x_1] \\
& x_2 & + a_{3:23}x_3 & = c_{3:21}y_1 & + c_{3:22}y_2 & + c_{3:23}y_3 & = d_{3:2} & = [d_2 - r_{21}d_{3:1}]/s_2 \\
r_{31}x_1 & + r_{32}x_2 & + s_3x_3 & = \cdots & & \cdots & \cdots & = d_{2:3} & = d_3 - r_{31}[d_{3:1} - x_1] - r_{32}[d_{2:2} - x_2] \\
& & x_3 & = c_{3:31}y_1 & + c_{3:32}y_2 & + c_{3:33}y_3 & = d_{3:3} & = [d_3 - r_{31}d_{3:1} - r_{32}d_{3:2}]/s_3
\end{array}
$$

$$
\begin{array}{rrrrrrrrrrrl}
x_1 & & & = c_{4:11}y_1 & + c_{4:12}y_2 & + c_{4:13}y_3 & = d_{4:1} & = d_{3:1} - a_{3:12}d_{3:2} - a_{3:13}d_{3:3} \\
& x_2 & & = c_{4:21}y_1 & + c_{4:22}y_2 & + c_{4:23}y_3 & = d_{4:2} & = d_{3:2} - a_{3:23}d_{3:3} \\
& & x_3 & = c_{4:31}y_1 & + c_{4:32}y_2 & + c_{4:33}y_3 & = d_{4:3} & = d_{3:3}
\end{array}
$$

TABLE II

Layout of Doolittle solution for the symmetric case. Coefficients not used further are indicated by ⋯ . Coefficients which can be filled by symmetry are indicated by —

$$
\begin{array}{rrrrrrrrrrrl}
a_{11}x_1 & + a_{12}x_2 & + a_{13}x_3 & = c_{11}y_1 & + c_{12}y_2 & + c_{13}y_3 & = d_1 \\
— & + a_{22}x_2 & + a_{23}x_3 & = c_{21}y_1 & + c_{22}y_2 & + c_{23}y_3 & = d_2 \\
— & — & + a_{33}x_3 & = c_{31}y_1 & + c_{32}y_2 & + c_{33}y_3 & = d_3
\end{array}
$$

$$
\begin{array}{rrrrrrrrrrrl}
s_1x_1 & + r_{21}x_2 & + r_{31}x_3 & = \cdots & & \cdots & & \cdots & = d_{2:1} & = d_1 \\
x_1 & + a_{3:12}x_2 & + a_{3:13}x_3 & = c_{3:11}y_1 & + c_{3:12}y_2 & + c_{3:13}y_3 & = d_{3:1} & = [d_1]/s_1 \\
& s_2x_2 & + r_{32}x_3 & = \cdots & & \cdots & \cdots & = d_{2:2} & = d_2 - r_{21}d_{3:1} \\
& x_2 & + a_{3:23}x_3 & = \cdots & + c_{3:22}y_2 & + c_{3:23}y_3 & = d_{3:2} & = [d_2 - r_{21}d_{3:1}]/s_2 \\
& & s_3x_3 & = \cdots & & \cdots & \cdots & = d_{2:3} & = d_3 - r_{31}d_{3:1} - r_{32}d_{3:2} \\
& & x_3 & = \cdots & & \cdots & + c_{3:33}y_3 & = d_{3:3} & = [d_2 - r_{31}d_{3:1} - r_{32}d_{3:2}]/s_3
\end{array}
$$

$$
\begin{array}{rrrrrrrrrrrl}
x_1 & & & = c_{4:11}y_1 & + c_{4:12}y_2 & + c_{4:13}y_3 & = d_{4:1} & = d_{3:1} - a_{3:12}d_{4:2} - a_{3:12}d_{4:3} \\
& x_2 & & = — & + c_{4:22}y_2 & + c_{4:23}y_3 & = d_{4:2} & = d_{3:2} - a_{3:23}d_{4:3} \\
& & x_3 & = — & — & + c_{4:33}y_3 & = d_{4:3} & = d_{3:3}
\end{array}
$$

Then

$$\begin{aligned} B &= A - R[A_3 - I] \\ &= A - R[(I + S^{-1}T) - I) && \text{by (4.5),} \\ &= A - RS^{-1}T \\ &= R + S + T. && \text{by (3.5) and (3.6).} \end{aligned} \tag{5.2}$$

Therefore we see that the prediagonal coefficients of the x's in (5.1) are the elements of R and that the diagonal coefficients are the elements of S. The rest of the coefficients in this set of equations are not needed in the calculations and have been indicated by dots in Table I.

6. The symmetric case. If the A matrix is symmetric, advantage can be taken of the fact that the B matrix is also symmetric. Therefore

$$S + T = S + R' \tag{6.1}$$

and the elements of S and R' can be written down just before the division by s_{ii} in the calculation of the A_3 matrix:

$$\begin{aligned} A_3 = (I + S^{-1}T) &= S^{-1}(S + T) && \text{by (4.5)} \\ &= S^{-1}(S + R'). \end{aligned} \tag{6.2}$$

The layout of the work is given in Table II.

If the C_4 matrix is symmetric, the prediagonal elements of C_4 can be entered by symmetry. Therefore it is not only unnecessary to calculate the prediagonal elements of C_4, but we can also omit the prediagonal elements of C_3. Note that in this case C_4 must be calculated from the right to the left as well as from the bottom up.

The most important case where it is known in advance that C_4 is symmetric is the determination of the inverse of a symmetric matrix. Then $C = I$ and $C_4 = A^{-1}$. Also the postdiagonal elements of C_3 are all zero so that the only elements of C_3 which have to be calculated are the diagonal elements. These are the reciprocals of the s_{ii}'s.

A case where C_4 is symmetric though $C \neq I$ will appear in a subsequent paper.

7. Norms. In order to state the conditions for error control in a matrix calculation, a concept of the norm or the absolute value of a matrix is necessary. In this paper the norm will be defined as the square root of the sum of the squares of the elements of the matrix. That is

$$N(F) = \sqrt{\Sigma_i \Sigma_j (f_{ij})^2}. \tag{7.1}$$

The two basic inequalities satisfied by the norm are

$$N(F + G) \leq N(F) + N(G) \tag{7.2}$$

and

$$N(FG) \leq N(F)N(G). \tag{7.3}$$

All other properties of the norm are derived from these.

For future reference we list the following norm relations:

$$\begin{aligned} N[(F)(I+G)] &= N(F+FG) \\ &\leq N(F) + N(F)N(G) \\ &\leq N(F)[1+N(G)]. \end{aligned} \tag{7.4}$$

If $N(F) < 1$ we have

$$\begin{aligned} N[(I+F)^{-1} - I] &= N[I - F + F^2 - \cdots - I] \\ &\leq N(F) + [N(F)]^2 + [N(F)]^3 + \cdots \\ &\leq \frac{N(F)}{1 - N(F)}. \end{aligned} \tag{7.5}$$

If $N(G - I) < 1$, (7.5) becomes

$$1 + N(G^{-1} - I) \leq \frac{1}{1 - N(G - I)}. \tag{7.6}$$

When $N(F - I) < 1$

$$\begin{aligned} N(F^{-1}G) &\leq N[I + (F^{-1} - I)][G] \\ &\leq [1 + N(F^{-1} - I)]\,N(G) \\ &\leq \frac{N(G)}{1 - N(F - I)} \qquad \text{by (7.6)} \end{aligned} \tag{7.7}$$

8. Error matrix and error norm. We shall also need a formal statement as to what we mean by error and we need a measure of the errors.

By an error matrix we mean the matrix whose elements consist of the differences between the value of the matrix elements as actually calculated and the true value of the matrix elements which would have been obtained if all calculations had been made exactly without any rounding or other approximations. The fundamental relation for the error matrix, $E[f(G)]$, of a function, $f(G)$, of G is

$$E[f(G)] = f[G + E(G)] - f(G). \tag{8.1}$$

If each element of a matrix is calculated to q decimal places and the matrix has p rows and p columns, the maximum rounding error introduced in any element is $.5 \times 10^{-q}$. The norm of the error introduced by rounding is less than

$$\begin{aligned} NE_1 &= \sqrt{p^2[.5 \times 10^{-q}]^2} \\ &= .5 \times 10^{-q} \times p. \end{aligned} \tag{8.2}$$

For triangular matrices

$$NE_2 = .5 \times 10^{-q}\sqrt{p(p+1)/2}. \tag{8.3}$$

For one column matrices

$$NE_3 = .5 \times 10^{-q}\sqrt{p}. \tag{8.4}$$

For future reference the following formulas for error norms are listed:

$$NE(F+G) \leq NE(F) + NE(G). \tag{8.5}$$

$$\begin{aligned} NE(FG) &\leq NE(F)N[G + E(G)] + NE(G)N(F) \\ &\leq NE(F)N(G) + NE(G)N(F) + NE(F)NE(G). \end{aligned} \tag{8.6}$$

$$\begin{aligned} NE[F(I+G)] &\leq NE(F)[1 + N[G + E(G)] + NE(G)N(F) \\ &\leq NE(F) + NE(F)N(G) + NE(G)N(F) \\ &\qquad + NE(F)NE(G). \end{aligned} \tag{8.7}$$

If $N(F-I) + NE(F) < 1$,

$$\begin{aligned} NE(F^{-1}) &= NE[I + (F-I)]^{-1} \\ &= N\left[\frac{1}{I + [F + E(F) - I]} - \frac{1}{I + (F-I)}\right] \\ &= N\{I + [F + E(F) - I]\}^{-1}\{I + (F-I)\}^{-1}\{E(F)\} \\ &\leq \frac{NE(F)}{\{1 - N[F + E(F) - I]\}\{1 - N(F-I)\}} \qquad \text{by (7.7),} \\ &\leq \frac{NE(F)}{[1 - N(F-I) - NE(F)][1 - N(F-I)]}. \end{aligned} \tag{8.8}$$

If $N(F-I) + NE(F) < 1$,

$$\begin{aligned} NE(F^{-1}G) &= N\left[\frac{G + E(G)}{I + [F + E(F) - I]} - \frac{G}{I + (F-I)}\right] \\ &= N\left[\frac{[E(G)]\,[I + (F-I)] - [E(F)]\,[G]}{\{I + [F + E(F) - I]\}\{I + (F-I)\}}\right] \\ &\leq \frac{NE(G) + \{[NE(F)]/[1 - N(F-I)]\}\{N(G)\}}{1 - N[F + E(F) - I]} \qquad \text{by (7.7).} \end{aligned} \tag{8.9}$$

9. Certain maxima. The R, S, and T matrices have no non-zero elements in common so that

$$\begin{aligned} N(B-I) &= N(R + S - I + T) \\ &= \sqrt{[N(R)]^2 + [N(S-I)]^2 + [N(T)]^2}. \end{aligned} \tag{9.1}$$

Similarly

$$NE(B) = \sqrt{[NE(R)]^2 + [NE(S)]^2 + [NE(T)]^2}. \tag{9.2}$$

The developments of the following formulas for certain maxima are not given here since they involve only well known calculus principles. It is understood that these inequalities hold whenever the quantities involved exist. Usually the maxima are given subject to the condition $N(T) = N(R)$ as well as for the unrestricted case where $N(T)$ may be zero.

$$\max \frac{N(R)N(T)}{1 - N(S - I)} = 1 - \sqrt{1 - [N(B - I)]^2}. \tag{9.3}$$

$$\max \frac{N(R)}{1 - N(S - I)} = \frac{N(B - I)}{\{1 - [N(B - I)]^2\}^{1/2}} \quad \text{if } N(T) = 0, \tag{9.4}$$

$$= \frac{N(B - I)}{\sqrt{2}\sqrt{1 - [N(B - I)]^2}} \quad \text{if } N(T) = N(R). \tag{9.5}$$

$$\max [N(R)NE(T) + N(T)NE(R)] = \sqrt{[N(B - I)]^2 - [N(S - I)]^2} \times \sqrt{[NE(B)]^2 - [NE(S)]^2}. \tag{9.6}$$

Any substitution satisfying the relation

$$\frac{NE(R)}{NE(T)} = \frac{N(T)}{N(R)} \tag{9.7}$$

will cause (9.6) to attain its maximum.

$$\max[NE(R) + kNE(S)] = \sqrt{\tfrac{1}{2} + k^2}\, NE(B) \quad \text{if } NE(R) = NE(T), \tag{9.8}$$

$$= \sqrt{1 + k^2}\, NE(B) \quad \text{if } NE(T) = 0. \tag{9.8.a}$$

$$\max \frac{k + N(R)}{1 - N(S - I)} = k + \frac{[N(B - I)]^2}{K} \tag{9.9}$$

where K is the root of

$$[1 + k^2]K^2 + 2[k\{N(B - \mathrm{I})\}^2]K + \{[N(B - I)]^4 - [N(B - I)]^2\} = 0. \tag{9.10}$$

10. Errors in the Doolittle method. In all that follows, we shall assume that $N(A - I)$ is small enough so that the "divisions" are permissible. First let us examine in the multipliers $B = R + S + T$. By (3.6)

$$B - I = (A - I) - RS^{-1}T. \tag{10.1}$$

Therefore

$$N(B - I) \leq N(A - I) + \frac{N(R)N(T)}{1 - N(S - 1)} \quad \text{by (7.3) and (7.7),}$$

$$\leq N(A - I) + 1 - \sqrt{1 - [N(B - I)]^2} \quad \text{by (9.3)} \tag{10.2}$$

Remembering that A has no error and letting NE_1 be the rounding error norm introduced in writing down the elements, we have

$$(10.3)\quad NE(B) \leq NE_1 + \frac{NE(R)N(T) - NE(T)N(R) + NE(R)NE(T)}{1 - N[S + E(S) - I]} + \frac{NE(S)N(T)N(R)}{\{1 - N[S + E(S) - I]\}\{1 - N(S - I)\}} \quad \text{by (8.9) and (8.6).}$$

We are interested only in the range of values where the errors are small. Therefore we shall ignore second order errors. Except for such errors,

$$(10.4)\quad NE(B) \leq NE_1 + \frac{N(T)}{1 - N(S - I)} NE(R) + \frac{N(R)}{1 - N(S - I)} NE(T) + \frac{N(T)N(R)}{[1 - N(S - I)]^2} NE(S).$$

The last term will be largest when $N(T) = N(R)$. Therefore the sum of the second and third will be largest when $NE(R) = NE(T)$ by (9.7).

$$(10.5)\quad NE(B) \leq NE_1 + 2\frac{N(R)}{1 - N(S - I)} NE(R) + \left[\frac{N(R)}{1 - N(S - I)}\right]^2 NE(S).$$

By (9.8) we now obtain

$$(10.6)\quad NE(B) \leq NE_1 + K\sqrt{2 + K^2}\, NE(B) \leq \frac{NE_1}{1 - K\sqrt{2 + K^2}}$$

where

$$K = \max \frac{N(R)}{1 - N(S - I)} = \frac{N(B - I)}{\sqrt{2}\sqrt{1 - [N(B - I)]^2}} \quad \text{by (9.5).}$$

Actually in practice we introduce the rounding error in $(S^{-1}T)$ instead of in T as assumed above. Our assumption is conservative since the division by S magnifies any error in T.

Next consider the errors in the C_3 (or D_3) matrix. From (4.2)

$$(10.7)\quad \begin{aligned} C_3 &= S^{-1}(C - RC_3) \\ &= S^{-1}C - S^{-1}RC_3 \\ &= (I + S^{-1}R)^{-1}S^{-1}C \quad \text{by (2.5) and (2.6),} \\ &= [S(I + S^{-1}R)]^{-1}C \end{aligned}$$

$$(10.8)\quad = (S + R)^{-1}C.$$

Therefore

$$N(C_3) \leq \frac{N(C)}{1 - N(R + S - I)} \quad \text{by (7.7),} \tag{10.9}$$

$$\leq \frac{N(C)}{1 - N(R) - N(S - I)}. \tag{10.10}$$

From (10.7), remembering that C has no error, and letting the rounding error be NE_3,

$$\begin{aligned} NE(C_3) &\leq NE_3 + \frac{NE(R)N(C_3) + NE(C_3)N[R + E(R)]}{1 - N[S + E(S) - I]} \\ &\quad + \frac{NE(S)[N(C) + N(R)N(C_3)]}{\{1 - N[S + E(S) - I]\}\{1 - N(S - I)\}} \quad \text{by (7.7) and (7.3),} \\ &\leq \frac{\dfrac{NE(R) + NE(S)}{1 - N(R) - N(S - I)} N(C) + \{NE_3\}\{1 - N[S + E(S) - I]\}}{1 - N[R + E(R)] - N[S + E(S) - I]} \end{aligned} \tag{10.11}$$

since

$$\begin{aligned} &\frac{N(C) + N(R)N(C_3)}{1 - N(S - I)} \\ &\quad \leq \frac{N(C)}{1 - N(S - I)}\left[1 + \frac{N(R)}{1 - N(R) - N(S - I)}\right] \quad \text{by (10.10),} \\ &\quad \leq \frac{N(C)[1 - N(S - 1)]}{[1 - N(S - I)][1 - N(R) - N(S - I)]} \end{aligned} \tag{10.12}$$

and since transferring the terms in $NE(C_3)$ to the left requires that we divide through by

$$1 - \frac{N[R + E(R)]}{1 - N[S + E(S) + I]} = \frac{1 - N[R + E(R)] - N[S + E(S) - I]}{1 - N[S + E(S) - I]}. \tag{10.13}$$

Again we can ignore second order errors. Taking maxima by (9.8.a) gives

$$NE(C_3) \leq \frac{\dfrac{\sqrt{2}\, NE(B)N(C)}{1 - \sqrt{2}\, N(B - I)} + NE_3}{1 - \sqrt{2}\, N(B - I)} \tag{10.14}$$

and

$$\frac{NE(C_3)}{N(C)} \leq \frac{\sqrt{2}\, NE(B)}{[1 - \sqrt{2}\, N(B - I)]^2} + \frac{[NE_3]/[N(C)]}{1 - \sqrt{2}\, N(B - I)}. \tag{10.15}$$

Thus we see that the proportionate error in C_3 is made up of two parts: the first due to the rounding errors in the multipliers as given by the first term and the second due to the proportionate rounding error introduced in calculating C_3.

Finally we have the errors in the C_4 (or D_4) matrix. Since

$$C_4 = A^{-1}C \tag{10.16}$$

we have

$$N(C_4) \leq \frac{N(C)}{1 - N(A - I)}. \tag{10.17}$$

By (4.6)

$$C_4 = C_3 - (S^{-1}T)C_4. \tag{10.18}$$

If we let NE_4 be the rounding error introduced in this step

$$\begin{aligned} NE(C_4) &\leq NE(C_3) + NE_4 + \frac{NE(T)N(C_4) + NE(C_4)N[T + E(T)]}{1 - N[S + E(S) - I]} \\ &\quad + \frac{NE(S)N(T)N(C_4)}{\{1 - N[S + E(S) - I]\}\{1 - N(S - I)\}} \\ &\leq \frac{\{NE(C_3) + NE_4\}\{1 - N[S + E(S) - I]\} + \left[NE(T) + \frac{N(T)NE(S)}{1 - N(S - I)}\right]\left[\frac{N(C)}{1 - N(A - I)}\right]}{1 - N[T + E(T)] - N[S + E(S) - I]} \end{aligned} \tag{10.19}$$

by (10.17) and relations similar to (10.13). We now ignore second order errors and take maxima by (9.4) and (9.8.a):

$$\begin{aligned} \frac{NE(C_4)}{N(C)} &\leq \frac{\frac{NE(C_3)}{N(C)} + \frac{(NE_4)}{N(C)} + \frac{\sqrt{1 + K^2}\,NE(B)}{1 - N(A - I)}}{1 - \sqrt{2}\,N(B - I)} \\ &\quad \left[\frac{1}{[1-N(A-I)]\sqrt{1-[N(B-I)]^2}} + \frac{\sqrt{2}}{[1-\sqrt{2}\,N(B-I)]^2}\right] NE(B) \\ &\leq \frac{+ \frac{[NE_3]/[N(C)]}{1-\sqrt{2}\,N(B-I)} + \frac{NE_4}{N(C)}}{1 - \sqrt{2}\,N(B - I)} \end{aligned} \tag{10.20}$$

where

$$K = \max \frac{N(T)}{1 - N(S - I)} = \frac{N(B - I)}{\sqrt{1 - [N(B - I)]^2}} \quad \text{by (9.4),} \tag{10.21}$$

and

$$\sqrt{1 + K^2} = 1/\sqrt{1 - [N(B - I)]^2}. \tag{10.22}$$

If A is symmetric, $NE(B - I)$ remains unchanged but $NE(C_4)$ can be somewhat strengthened through the use of (9.5) and (9.8) instead of (9.4) and (9.8.a).

The result is

$$\frac{NE(C_4)}{N(C)} \leq \left[\frac{1}{[1 - N(A - I)]\sqrt{2}\sqrt{1 - [N(B - I)]^2}} + \frac{\sqrt{3/2}}{[1 - \sqrt{3/2}\,N(B - I)]^2}\right]\left[\frac{NE(B)}{1 - \sqrt{3/2}\,N(B - I)}\right] + \frac{1}{[1 - \sqrt{3/2}\,N(B - I)]^2}\,\frac{NE_3}{N(C)} + \frac{1}{1 - \sqrt{3/2}\,N(B - I)}\,\frac{NE_4}{N(C)}. \tag{10.23}$$

If C is approximately equal to I, a better formula is obtained by substituting $(I + C_1)$ for C. The final formulas then are identical to (10.20) and (10.23) with the substitution of $1 + N(C_1)$ for $N(C)$. Similarly if $C = I$ so that $C_4 = A^{-1}$, $N(C) = 1$ should be substituted in (10.20) and (10.23).

If A is symmetric and if $C = I$ so that $C_4 = A^{-1}$, the prediagonal elements of C_4 are filled in by symmetry as in Table II instead of being calculated directly. This complicates the analysis of error relations. The following inequality gives the error limit for the diagonal and postdiagonal elements of (A^{-1}). We have indicated these elements by F.

$$NE(F) \leq \frac{\sqrt{[3/2] - N(A - I) + [N(B - I)]^2/K}}{[1 - N(A - I)]\,[1 - \sqrt{2}\,N(B - I)]}\,NE(B) + \frac{NE_5}{1 - \sqrt{2}\,N(B - I)} \tag{10.24}$$

where K is the root of (9.10) when $k = 1 - N(A - I)$ and $NE_5 = 0.5 \times 10^{-q} \times \sqrt{p(p + 1)/2}$ by (8.3).

11. Results. Given the matrix A, we subtract one from each element on the principal diagonal to obtain the matrix $(A - I)$. By the norm of $(A - I)$ we mean the square root of the sum of the squares of the elements of $(A - I)$. We shall now show that a Doolittle process such as outlined in Table I is in a state of error control if the norm of $(A - I)$ is less than 0.35:

1. $N(A - I) \leq 0.35$
2. $N(B - I) \leq 0.4642$ by (10.2).
3. $NE(B) \leq 1.09\ p$ by (10.6) and (8.2)

if the maximum rounding error in any element is 0.5 and A has p rows. Thus no element of the multiplying matrices R, S, or T can have an error of greater than this amount.

4. $NE(C_4) \leq (44 \times p \times 10^{-q})N(C) + (6 \times p \times 10^{-r})$ by (10.20) and (8.2) where q decimal places are carried on the left and r on the right. Thus if the decimal point in C is shifted so that $N(C) \leq 1$, the error in any element of C_4 can

not amount to more than three significant figures if the same number of decimal places are carried on both the left and on the right (four significant figures for 21 to 200 rows).

5. $NE(A^{-1}) \leq$ three significant figures by substituting $N(C) = 1$ since $C = I$.

As we let $N(A - I)$ become larger than 0.35, the maximum errors indicated by our formulas rapidly become very large. In fact they become infinite if $N(A - I) = 0.414$.

Since for more than four equations the above formulas show errors in the second decimal place no matter how small $N(A - I)$ is, it is suggested that as a general practice:

1. The problem be arranged so that $N(A - I) \leq 0.35$.
2. The decimal points in C be shifted so $N(C) < 1$.
3. Three extra decimal places be carried in the calculations.

12. Preliminary adjustments. The requirement that $N(A - I)$ should be less than 0.35 is not normally met in practical problems. If, however, an approximation to A^{-1} is available, the problem can almost always be rearranged to satisfy this condition.

Thus if we are solving the equations such as given in Table I,

$$AX = CY = D, \tag{12.1}$$

we are perfectly free to multiply through by any non-singular matrix F without disturbing the solution:

$$(FA)X = (FC)Y = (FD). \tag{12.2}$$

Now if F is a sufficiently close approximation to A^{-1}, FA will be almost equal to I. Therefore $N(FA - I)$ will be less than 0.35 and a Doolittle solution of (12.2) will be in a state of error control.

Similarly for the inversion of A, we can apply the Doolittle process to the pair of matrices

$$FA : F$$

just as easily as to the pair

$$A : I$$

since

$$\begin{aligned}(FA)^{-1} \,||\, FA : F \,|| &= A^{-1}F^{-1} \,||\, FA : F \,|| \\ &= ||\, A^{-1}(F^{-1}F)A : A^{-1}(F^{-1}F) \,|| \\ &= ||\, A^{-1}A : A^{-1} \,|| \\ &= ||\, I : A^{-1} \,||.\end{aligned}$$

Thus by taking F as a sufficiently close approximation to A^{-1}, we can bring an inversion calculation into a state of error control.

The computor should be cautioned that the multiplication by F must be exact and that no rounding is allowable in this step. Our formulas assumed that we started with matrices free of error.

13. Further work. The principles used in this paper can be applied to the task of developing calculation routines which will be in a state of error control regardless of the size of $N(A - I)$. Enough work has been done to see that such routines do exist and do not involve prohibitive labor. The author expects that the most efficient routine will be to use these more elaborate methods to obtain an F such that $N(FA - I) \leq 0.35$ and then to use the normal Doolittle methods as outlined in section 12.

REFERENCES

The writer wishes to acknowledge particularly his debt to Prof. Hotelling's very complete paper on interative methods and to Prof. Dwyer's papers on the Doolittle methods. These papers have such complete bibliographies, we shall not give further references here.

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INVERSE TABLES OF PROBABILITIES OF ERRORS OF THE SECOND KIND

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1. Introduction. The problem of testing linear hypotheses was discussed by Kolodziejczyk [1], and later in greater detail by Tang [2], who computed a table giving the probabilities of errors of the second kind P_{II} for a range of values of the two degrees of freedom f_1 and f_2, ($f_1 = 1(1)8, f_2 = 2, 4, 6(1)30, 60, \infty$)[1] and for two fixed levels $P_I = .01$ and $.05$ of the probability of errors of the first kind. These tables are in terms of a parameter φ, ($\varphi = 1(.5)3(1)8$) whose statistical significance, or rather that of

$$\lambda = (f_1 + 1)\varphi^2/2$$

is discussed in Tang's paper. · A restatement of the problem of testing linear hypotheses in a more canonical form, giving an interpretation of λ, will also be found in a recent paper by Wald [3].

Professor Neyman has felt for some time that a table giving $\varphi = \varphi(a, b, \alpha, \beta)$ as a function of the two degrees of freedom $f_1 = 2a$, and $f_2 = 2b$, and of the two probability levels $\alpha = P_I$ and $\beta = 1 - P_{II}$ would be more useful for statistical purposes, where β is the probability of detecting the falsehood of the hypothesis tested. A paper by Professor Neyman explaining this point of view and giving applications of the present tables to some statistical problems will appear shortly. These tables were computed in the Statistical Laboratory of the University of California,[2] and give values of φ for the following range of parameters:

$$(\alpha, \beta) = (.01, .7), (.01, .8), (.05, .7), (.05, .8)$$

$$f_1 = 1(1)10, 12, 15, 20, 24, 30, 40, 60, 80, 120, \infty.$$

$$f_2 = 2(2)20, 24, 30, 40, 60, 80, 120, 240, \infty.$$

2. Analytic definitions. The statistical parameter

$$\lambda = \lambda(a, b, \alpha, \beta) = (a + \tfrac{1}{2})\,\varphi^2(a, b, \alpha, \beta) \tag{1}$$

can be thought of as an inverse function connected with the hypergeometric distribution. Inverse functions $y(\alpha)$, $u(a, \alpha)$ and $x(a, b, \alpha)$ of the better known normal, Gamma and Beta distributions respectively have all been tabulated,

[1] The notation $m = r(s)t$ is equivalent to $m = r, r + s, r + 2s, \ldots, t$.

[2] These tables were begun by Miss Leone Gintzler, and were carried on by Mark Eudy under a University of California Research Grant. The bulk of the computing was done, however, by the author and by Mrs. Julia Robinson under a grant of the American Philosophical Society.

and are sometimes called "percentage points" of the distribution. To begin with the simplest, the normal distribution, we may define $y(\alpha)$ as the solution of

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y(\alpha)} e^{-x^2/2}\, dx = \alpha. \tag{2}$$

This function has been recently tabulated by Truman Lee Kelley [4] for $\alpha = p = 0(.0001)$ 1 to 8 decimal places.

The function $u(a, \alpha)$ is the solution of

$$\frac{1}{\Gamma(a)} \int_0^{u(a,\alpha)} t^{a-1} e^{-t}\, dt = \Gamma_u(a)/\Gamma(a) = \alpha. \tag{3}$$

This is connected with the well known percentage points of the χ^2_ν distribution with $\nu = 2a$ degrees of freedom as follows:

$$\chi^2_\nu(\alpha) = 2u(a, 1 - \alpha). \tag{4}$$

Catherine Thomson [5] has tabulated $\chi^2_\nu(\alpha)$ for $\alpha = .005, .01, .025, .05, .1, .25, .5, .75, .9, .95, .975, .99, .995$, and for $\nu = 1(1)30(10)100$. She has also tabulated [6] the corresponding parameter $x = x(a, b, \alpha)$ of the Beta distribution with $\nu_1 = 2b$, $\nu_2 = 2a$ degrees of freedom defined by

$$\frac{1}{\mathrm{B}(a, b)} \int_0^{x(a,b,\alpha)} t^{a-1}(1 - t)^{b-1}\, dt = \frac{\mathrm{B}_x(a, b)}{\mathrm{B}(a, b)} = \alpha \tag{5}$$

for $\alpha = .005, .01, .025, .05, .1, .25$ and $.5$ and $\nu_1 = 1(1)10, 12, 15, 20, 24, 30, 40, 60, 120$, $\nu_2 = 1(1)30, 40, 60, 120, \infty$ to five significant places.

Similarly $\lambda(a, b, \alpha, \beta)$ can be defined as the solution of

$$\frac{1}{\mathrm{B}(a, b)} \int_0^{x(a,b,1-\alpha)} e^{-\lambda(1-t)} t^{a-1}(1 - t)^{b-1} F(-b, a, -\lambda t)\, dt = 1 - \beta \tag{6}$$

where

$$F(\gamma, \delta, z) = 1 + \frac{\gamma}{\delta} z + \frac{\gamma(\gamma + 1)}{\delta(\delta + 1)} z^2 + \cdots$$

is the confluent hypergeometric function.

3. Limiting cases. It is well known that as a tends to infinity

$$\frac{\chi^2_\nu(\alpha)}{\nu} = \frac{u(a, 1 - \alpha)}{a} = 1 + \frac{y(\alpha)}{\sqrt{a}} + 0\left(\frac{1}{a}\right). \tag{7}$$

There are many approximations [7] to χ^2. In a recent paper Peiser [8] gave a rigorous derivation of an asymptotic formula for χ^2.

Similarly, the limiting cases of $x(a, b, \alpha)$ as a and b tend to infinity are known

TABLE OF φ

Level of Significance, $\alpha = .05$. Probability of Detecting the Falsehood of the Hypothesis Tested, $\beta = .7$

f_2 \ f_1	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞	240//f_2
2	3.438	3.920	4.141	4.267	4.350	4.408	4.451	4.484	4.510	4.532	4.564	4.598	4.632	4.650	4.667	4.685	4.704	4.722	4.740	
4	2.341	2.418	2.436	2.441	2.442	2.442	2.441	2.440	2.439	2.438	2.437	2.434	2.432	2.43	2.429	2.43	2.426	2.424	2.421	
6	2.104	2.091	2.059	2.031	2.010	1.992	1.978	1.966	1.956	1.948	1.935	1.921	1.905	1.90	1.888	1.88	1.870	1.860	1.850	
8	2.003	1.950	1.895	1.852	1.818	1.792	1.771	1.754	1.739	1.727	1.707	1.685	1.662	1.65	1.636	1.62	1.608	1.593	1.576	
10	1.948	1.872	1.803	1.751	1.710	1.679	1.653	1.632	1.614	1.599	1.575	1.548	1.519	1.50	1.487	1.47	1.451	1.431	1.411	
12	1.913	1.823	1.745	1.687	1.641	1.606	1.577	1.553	1.533	1.516	1.488	1.457	1.424	1.41	1.386	1.37	1.344	1.321	1.297	
14	1.888	1.789	1.705	1.642	1.593	1.554	1.523	1.497	1.475	1.457	1.426	1.393	1.356	1.34	1.314	1.29	1.267	1.240	1.212	
16	1.871	1.764	1.675	1.609	1.557	1.517	1.483	1.456	1.432	1.413	1.380	1.344	1.304	1.28	1.259	1.23	1.206	1.178	1.146	
18	1.857	1.744	1.653	1.584	1.530	1.487	1.453	1.424	1.400	1.378	1.344	1.306	1.264	1.24	1.215	1.19	1.158	1.127	1.093	
20	1.846	1.729	1.635	1.563	1.508	1.464	1.428	1.398	1.373	1.351	1.316	1.276	1.231	1.21	1.180	1.15	1.120	1.086	1.048	12
24	1.831	1.707	1.608	1.534	1.476	1.430	1.392	1.360	1.333	1.310	1.273	1.230	1.182	1.16	1.13	1.09	1.06	1.02	.978	10
30	1.815	1.685	1.582	1.505	1.444	1.396	1.356	1.322	1.294	1.269	1.229	1.183	1.131	1.10	1.070	1.03	.996	.951	.902	8
40	1.80	1.66	1.56	1.48	1.41	1.36	1.32	1.28	1.25	1.23	1.19	1.14	1.08	1.05	1.01	.97	.93	.88	.816	6
60	1.786	1.643	1.532	1.447	1.381	1.328	1.284	1.247	1.215	1.187	1.141	1.088	1.026	.99	.951	.90	.854	.792	.714	4
80	1.78	1.63	1.52	1.43	1.37	1.31	1.27	1.23	1.20	1.17	1.12	1.06	1.00	.96	.92	.87	.81	.75	.653	3
120	1.77	1.62	1.51	1.42	1.35	1.29	1.25	1.21	1.18	1.15	1.10	1.04	.97	.93	.87	.83	.77	.692	.577	2
240	1.76	1.61	1.50	1.41	1.33	1.28	1.23	1.19	1.16	1.13	1.07	1.01	.94	.90	.85	.79	.72	.63	.473	1
∞	1.757	1.602	1.483	1.392	1.320	1.262	1.213	1.170	1.135	1.104	1.051	.988	.913	.87	.815	.75	.673	.557	0	0

TABLE OF φ

Level of Significance, $\alpha = .05$. Probability of Detecting the Falsehood of the Hypothesis Tested, $\beta = .8$

f_2 \ f_1	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞	240/f_2
2	3.898	4.558	4.814	4.961	5.057	5.125	5.175	5.213	5.244	5.269	5.307	5.346	5.386	5.406	5.427	5.447	5.469	5.490	5.511	
4	2.659	2.736	2.752	2.755	2.755	2.754	2.752	2.750	2.749	2.747	2.745	2.742	2.739	2.74	2.735	2.73	2.730	2.728	2.725	
6	2.381	2.351	2.309	2.274	2.248	2.226	2.209	2.195	2.183	2.174	2.158	2.141	2.122	2.11	2.103	2.09	2.081	2.069	2.058	
8	2.263	2.188	2.119	2.066	2.026	1.995	1.970	1.949	1.932	1.918	1.894	1.869	1.841	1.83	1.811	1.80	1.779	1.761	1.742	
10	2.199	2.098	2.014	1.951	1.903	1.865	1.835	1.810	1.789	1.771	1.743	1.712	1.678	1.66	1.640	1.62	1.599	1.577	1.553	
12	2.159	2.041	1.947	1.877	1.824	1.782	1.748	1.720	1.696	1.676	1.644	1.609	1.570	1.55	1.526	1.50	1.478	1.452	1.423	
14	2.131	2.002	1.901	1.826	1.769	1.723	1.687	1.656	1.631	1.609	1.574	1.535	1.493	1.47	1.444	1.42	1.390	1.360	1.328	
16	2.110	1.974	1.868	1.789	1.730	1.681	1.642	1.610	1.583	1.560	1.522	1.481	1.434	1.41	1.382	1.35	1.322	1.289	1.253	
18	2.095	1.952	1.842	1.760	1.698	1.647	1.607	1.574	1.545	1.521	1.482	1.438	1.389	1.36	1.333	1.30	1.268	1.233	1.193	
20	2.083	1.935	1.822	1.737	1.673	1.621	1.580	1.545	1.516	1.490	1.449	1.404	1.352	1.32	1.294	1.26	1.225	1.186	1.143	12
24	2.065	1.910	1.792	1.704	1.636	1.583	1.539	1.501	1.471	1.444	1.401	1.352	1.297	1.27	1.23	1.20	1.16	1.11	1.065	10
30	2.047	1.886	1.763	1.671	1.601	1.544	1.498	1.460	1.427	1.399	1.352	1.300	1.240	1.21	1.171	1.13	1.087	1.037	.979	8
40	2.03	1.86	1.73	1.64	1.57	1.51	1.46	1.42	1.38	1.35	1.30	1.25	1.18	1.15	1.11	1.06	1.01	.95	.885	6
60	2.014	1.838	1.706	1.607	1.531	1.469	1.418	1.375	1.339	1.307	1.254	1.194	1.124	1.08	1.039	.99	.930	.857	.773	4
80	2.01	1.83	1.69	1.59	1.51	1.45	1.40	1.35	1.32	1.28	1.23	1.17	1.09	1.05	1.00	.95	.88	.81	.705	3
120	2.00	1.81	1.68	1.58	1.50	1.43	1.38	1.33	1.30	1.26	1.21	1.14	1.06	1.02	.97	.91	.84	.750	.623	2
240	1.99	1.80	1.67	1.56	1.48	1.41	1.36	1.31	1.27	1.24	1.18	1.11	1.03	.98	.93	.87	.79	.68	.509	1
∞	1.981	1.792	1.654	1.545	1.462	1.395	1.339	1.292	1.251	1.215	1.155	1.084	.997	.95	.889	.82	.731	.603	0	0

TABLE OF φ

Level of Significance, $\alpha = .01$. Probability of Detecting the Falsehood of the Hypothesis Tested, $\beta = .8$

f_2 \ f_1	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞	240/f_2
2	8.965	10.326	10.943	11.297	11.527	11.689	11.809	11.901	11.974	12.034	12.125	12.219	12.314	12.363	12.412	12.462	12.512	12.563	12.613	
4	4.157	4.311	4.353	4.372	4.380	4.383	4.385	4.386	4.386	4.386	4.386	4.386	4.384	4.383	4.383	4.381	4.380	4.379	4.378	
6	3.371	3.335	3.282	3.239	3.205	3.179	3.157	3.140	3.125	3.113	3.093	3.072	3.050	3.037	3.023	3.012	2.998	2.984	2.969	
8	3.068	2.959	2.866	2.798	2.746	2.706	2.674	2.647	2.625	2.607	2.577	2.545	2.511	2.492	2.473	2.453	2.432	2.410	2.388	
10	2.910	2.763	2.649	2.566	2.503	2.455	2.416	2.384	2.358	2.335	2.299	2.260	2.218	2.195	2.171	2.146	2.120	2.092	2.063	
12	2.813	2.642	2.515	2.423	2.353	2.300	2.256	2.220	2.191	2.166	2.125	2.081	2.032	2.006	1.978	1.950	1.919	1.886	1.851	
14	2.748	2.561	2.425	2.327	2.252	2.194	2.148	2.109	2.077	2.049	2.005	1.957	1.904	1.875	1.844	1.812	1.778	1.741	1.701	
16	2.701	2.503	2.359	2.257	2.179	2.118	2.069	2.028	1.994	1.965	1.918	1.866	1.809	1.778	1.745	1.710	1.672	1.632	1.588	
18	2.666	2.460	2.312	2.205	2.124	2.060	2.009	1.966	1.931	1.901	1.851	1.796	1.736	1.703	1.668	1.630	1.590	1.546	1.498	
20	2.638	2.426	2.274	2.164	2.080	2.014	1.962	1.918	1.881	1.850	1.798	1.741	1.679	1.644	1.607	1.567	1.524	1.477	1.425	12
24	2.60	2.38	2.22	2.10	2.02	1.95	1.89	1.85	1.81	1.77	1.72	1.66	1.59	1.56	1.52	1.47	1.42	1.37	1.312	10
30	2.559	2.328	2.164	2.046	1.955	1.884	1.826	1.778	1.737	1.702	1.644	1.580	1.507	1.47	1.421	1.37	1.321	1.261	1.193	8
40	2.52	2.28	2.11	1.99	1.89	1.82	1.76	1.71	1.67	1.63	1.57	1.50	1.42	1.38	1.33	1.27	1.21	1.14	1.064	6
60	2.485	2.236	2.062	1.935	1.837	1.760	1.696	1.643	1.598	1.559	1.494	1.420	1.335	1.29	1.232	1.17	1.102	1.019	.917	4
80	2.47	2.21	2.04	1.91	1.81	1.73	1.66	1.61	1.56	1.52	1.46	1.38	1.29	1.24	1.18	1.12	1.04	.95	.830	3
120	2.45	2.19	2.01	1.88	1.78	1.70	1.63	1.58	1.53	1.49	1.42	1.34	1.25	1.20	1.13	1.06	.98	.875	.727	2
240	2.43	2.17	1.99	1.86	1.75	1.67	1.60	1.55	1.50	1.46	1.38	1.30	1.20	1.15	1.08	1.00	.91	.79	.587	1
∞	2.417	2.150	1.966	1.830	1.726	1.642	1.572	1.514	1.464	1.422	1.346	1.256	1.158	1.10	1.026	.94	.839	.687	0	0

TABLE OF φ

Level of Significance, $\alpha = .01$. Probability of Detecting the Falsehood of the Hypothesis Tested, $\beta = .7$

f_2 \ f_1	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞	240/f_2
2	7.746	8.922	9.455	9.761	9.960	10.094	10.204	10.282	10.346	10.398	10.476	10.557	10.640	10.681	10.724	10.767	10.810	10.855	10.900	
4	3.714	3.858	3.900	3.917	3.923	3.929	3.931	3.932	3.933	3.933	3.934	3.933	3.932	3.932	3.931	3.930	3.929	3.928	3.928	
6	3.037	3.015	2.972	2.935	2.907	2.884	2.865	2.850	2.838	2.827	2.810	2.790	2.771	2.761	2.750	2.738	2.726	2.713	2.700	
8	2.773	2.686	2.608	2.548	2.503	2.468	2.440	2.416	2.398	2.381	2.356	2.327	2.297	2.281	2.264	2.246	2.227	2.208	2.188	
10	2.634	2.512	2.415	2.343	2.289	2.246	2.212	2.184	2.160	2.141	2.109	2.074	2.037	2.016	1.995	1.973	1.950	1.925	1.899	
12	2.548	2.406	2.296	2.216	2.155	2.108	2.070	2.038	2.012	1.990	1.954	1.914	1.871	1.848	1.824	1.798	1.770	1.740	1.710	
14	2.490	2.334	2.216	2.130	2.064	2.013	1.972	1.938	1.910	1.886	1.847	1.804	1.756	1.730	1.703	1.674	1.645	1.611	1.575	
16	2.448	2.282	2.158	2.068	1.999	1.945	1.901	1.865	1.834	1.807	1.768	1.722	1.671	1.643	1.614	1.582	1.548	1.512	1.473	
18	2.417	2.243	2.114	2.020	1.949	1.893	1.847	1.810	1.778	1.752	1.708	1.659	1.605	1.577	1.545	1.511	1.474	1.435	1.392	
20	2.392	2.212	2.080	1.984	1.910	1.852	1.805	1.767	1.734	1.706	1.660	1.610	1.553	1.525	1.489	1.453	1.414	1.372	1.326	12
24	2.356	2.167	2.030	1.930	1.853	1.79	1.74	1.70	1.67	1.64	1.59	1.54	1.48	1.44	1.40	1.36	1.32	1.28	1.223	10
30	2.321	2.124	1.982	1.877	1.797	1.734	1.682	1.639	1.603	1.571	1.520	1.462	1.397	1.36	1.321	1.28	1.230	1.176	1.115	8
40	2.29	2.08	1.94	1.83	1.74	1.68	1.62	1.58	1.54	1.50	1.45	1.39	1.32	1.28	1.24	1.19	1.13	1.07	.996	6
60	2.255	2.041	1.889	1.776	1.690	1.621	1.564	1.517	1.476	1.441	1.383	1.316	1.239	1.20	1.147	1.09	1.029	.954	.860	4
80	2.24	2.02	1.87	1.75	1.66	1.59	1.54	1.49	1.44	1.41	1.35	1.28	1.20	1.16	1.10	1.04	.97	.89	.780	3
120	2.22	2.00	1.84	1.73	1.64	1.57	1.51	1.46	1.41	1.38	1.32	1.24	1.16	1.11	1.05	.99	.91	.820	.684	2
240	2.21	1.98	1.82	1.70	1.61	1.54	1.48	1.43	1.38	1.35	1.28	1.21	1.12	1.07	1.01	.94	.85	.74	.560	1
∞	2.193	1.963	1.801	1.678	1.587	1.513	1.451	1.400	1.353	1.313	1.247	1.169	1.076	1.02	.956	.882	.785	.647	0	0

[9] and follow readily from (5), although no attempt has been made, as far as I know, to find better approximations. These limiting values are as follows:

$$\lim_{a\to\infty} a[1 - x(a, b, \alpha)] = u(b, 1 - \alpha) \tag{8}$$

$$\lim_{b\to\infty} bx(a, b, \alpha) = u(a, \alpha). \tag{9}$$

The two corresponding limiting cases for λ are not at first glance so symmetric. When a tends to infinity, we have from (1)

$$\lim_{a\to\infty} \frac{\lambda}{a} = \varphi^2 \tag{10}$$

while

$$\lim_{a\to\infty} F(-b, a, -a\varphi^2 t) = (1 + \varphi^2 t)^b. \tag{11}$$

Substituting these in (6) and letting $t = 1 - z/a(1 + \varphi^2)$ and passing to the limit we get with the help of (8)

$$\frac{1}{\Gamma(b)} \int_{(1+\varphi^2)u(b,\alpha)}^{\infty} e^{-z} z^{b-1}\, dz = 1 - \beta. \tag{12}$$

In other words

$$(1 + \varphi^2)u(b, \alpha) = u(b, \beta)$$

or

$$\varphi(\infty, b, \alpha, \beta) = \sqrt{\frac{u(b, \beta)}{u(b, \alpha)} - 1}. \tag{13}$$

This is the only case, except for $b = 1$ in which φ can be given explicitly. For $b = 1$, we have from (5)

$$x(a, 1, \alpha) = \sqrt[a]{\alpha}$$

and (6) can be easily integrated to give

$$\varphi(a, 1, \alpha, \beta) = \left[\log\left(\frac{1 - \alpha}{1 - \beta}\right) \Big/ \left(a + \frac{1}{2}\right)(1 - \sqrt[a]{1 - \alpha})\right]^{\frac{1}{2}}.$$

In all other cases it was found impractical to attempt an inversion of (6) to get φ.

When b becomes infinite (6) becomes with $t = z/b$, and with the help of (9),

$$e^{-\lambda} \int_0^{u(a,1-\alpha)} e^{-z} \left(\frac{z}{\lambda}\right)^{(a-1)/2} I_{a-1}(2\sqrt{\lambda z})\, dz = 1 - \beta \tag{14}$$

where

$$I_{a-1}(2\sqrt{\lambda z}) = \frac{(\lambda z)^{-(a-1)/2}}{\Gamma(a)} \lim_{b\to\infty} F\left(-b, a, -\frac{\lambda z}{b}\right) \tag{15}$$

is the Bessel function of a purely imaginary argument, which is usually defined by

$$I_n(x) = \sum_{\nu=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2\nu}}{\nu!\Gamma(n+\nu+1)}.$$

Expression (14) was also obtained for this limiting case of the hypergeometric distribution by Wishart [10]. This integral, however, does not give λ any more explicitly than the general integral (6), and since the calculation of λ increases in difficulty as a increases, an attempt was made to derive an approximate formula for $\varphi(a, \infty, \alpha, \beta)$ for large a. To this end an asymptotic formula [11] was developed for $I_n(nx)$, the principal term of which is

$$I_n(nx) = \frac{(1+x^2)^{1/4}}{\sqrt{2\pi n}} \left(\frac{\sqrt{1+x^2}-1}{x} e^{\sqrt{1+x^2}} \right)^n. \tag{16}$$

Substituting this into (14) with $x = 2\sqrt{\lambda z}/a - 1$ and $n = a - 1$ we get for large a

$$\frac{e^{-\lambda}}{\Gamma(a)} \int_0^{u(a,1-\alpha)} t^{a-1} e^{-t(1-\varphi^2+t\varphi^4/2a)}\, dt = 1 - \beta \tag{17}$$

If we assume that φ is sufficiently small to neglect the term in φ^4 we get as a first approximation

$$\frac{1}{\Gamma(a)} \int_0^{(1-\varphi^2)u(a,1-\alpha)} t^{a-1} e^{-t}\, dt = 1 - \beta \tag{18}$$

or

$$\varphi(a, \infty, \alpha, \beta) \sim \sqrt{1 - \frac{u(a, 1-\beta)}{u(a, 1-\alpha)}} \tag{19}$$

a formula very similar to (11). In fact since a is large this formula can be reduced one more step using (7). This gives

$$\lim_{a\to\infty} \sqrt{a}\varphi^2(a, \infty, \alpha, \beta) = y(\alpha) - y(\beta). \tag{20}$$

Similarly (13) becomes

$$\lim_{b\to\infty} \sqrt{a}\varphi^2(\infty, b, \alpha, \beta) = y(\alpha) - y(\beta).$$

If instead of neglecting the term in φ^4, we multiply it by the value of t at its upper limit, we get

$$\varphi^2 \sim \frac{1 - \sqrt{1 - 2[u(a, 1-\alpha) - u(a, 1-\beta)]/a}}{u(a, 1-\alpha)/a}. \tag{21}$$

Professor Neyman derived another approximation for $\varphi(a, \infty, \alpha, \beta)$ by assuming that the distribution (14) approaches a normal distribution for large a. He obtained:

$$\sqrt{a}\varphi^2 \sim y(\alpha) + \frac{y^2(\beta)}{\sqrt{a}} - y(\beta)\left(1 + \frac{2y(\alpha)}{\sqrt{a}} + \frac{y^2(\beta)}{a}\right)^{1/2}. \tag{22}$$

Both (21) and (22) obviously reduce to (20) in the limit. The following table shows the efficiency of these formulas for $a = 60$

	Table	(21)	(22)	(20)
$\alpha = .01$, $\beta = .8$	.687	.695	.668	.640
$\alpha = .01$, $\beta = .7$	.647	.642	.622	.607
$\alpha = .05$, $\beta = .8$	.603	.585	.593	.566
$\alpha = .05$, $\beta = .7$	.557	.540	.544	.529

A rigorous derivation of some such formula giving the actual order of approximation of φ would of course be of interest, but is likely to be quite complicated.

4. Calculation of tables. It is fairly obvious that the integral (6), although very useful theoretically is not well adapted to actual calculations. It can easily be integrated by parts to produce the infinite series.

$$e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!}\frac{B_{1-x}(a+i,b)}{B(a+i,b)} = e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!}\frac{B_x(b,a+i)}{B(a+i,b)} = 1-\beta. \tag{23}$$

This series can be used effectively for calculation purposes only if λ is comparatively small. If b is an integer, however, this series can be replaced by a finite series of b terms, which was also used by Tang [2] in calculating his tables. This series is as follows with $x = x(a, b, 1-\alpha) = x(b, a, \alpha)$:

$$e^{-\lambda x}(1-x)^{a+b-1}\sum_{i=0}^{b-1} T_i = 1-\beta \tag{24}$$

where

$$T_0 = 1, \quad T_1 = x[\lambda(1-x)+a+b-1]/(1-x)$$

and[3]

$$T_n = x\{[\lambda(1-x)+a+b-n]\,T_{n-1} + \lambda x T_{n-2}\}/n(1-x). \tag{25}$$

The subjoined tables can be thought of as inverses of Tang's tables, and could have been obtained from tables such as Tang's by inverse interpolation, had the interval of tabulation been sufficiently fine. The interval of tabulation of .5 for φ allowed only a crude approximation or trial value of φ, the corresponding probability was then calculated for this point, and then corrected with the help

[3] It will be noticed in comparing these formulas with those given by Tang, that x is used for $1-x$. This is done to conform with Miss Thomson's table for x.

of derivatives. In the beginning of the work, a recalculation was usually made for the corrected value of φ, and the tabulated value of φ was then obtained by inverse interpolation between these very close values. As the work progressed, and difference tables were calculated in several directions, the guesses improved considerably so that a correction could be made using the first derivative to give a tabulated value of φ correct to three decimal places. Such corrections never exceeded .004, and therefore it is hoped that the tables are correct to the last place. The derivative in question is given by

$$\frac{d\beta}{d\varphi} = (2a + 1)\varphi x(1 - x)^{a+b-1}e^{-\lambda x}T_{b-1}, \tag{26}$$

and was obtained as a by-product of the calculation of (24). This method was used for all finite a's and for all b's less than 30. For $b = 30$ or more it was found more expeditious to use the infinite series (23), about 20 terms of which sufficed.

The values of $x(a, b, 1 - \alpha) = x(b, a, \alpha)$ used in these calculations were obtained to five significant places from Miss Thomson's table [6] for $\alpha = .01$ and .05 with $\nu_1 = 2a$, $\nu_2 = 2b$.

No calculations were made for non-integer b's since for small values of b, λ was too large to make the infinite series (23) practicable, while for $b \geqq 7$, φ can be easily obtained by interpolation. The only available method for calculating φ for $b = 1/2, 3/2$ or $5/2$ would be by numerical integration of (6), which would be rather lengthy. Furthermore, the interest seems to be in large rather than small values of a and b.

5. Calculations for infinite cases. The case $a = \infty$ was readily disposed of using (13), as for $b = \infty$, the integral (14) was again integrated by parts to give

$$e^{-(\lambda+u)} \sum_{\nu=0}^{\infty} \left(\frac{u}{\lambda}\right)^{(a+\nu)/2} I_{a+\nu}(2\sqrt{u\lambda}) = 1 - \beta. \tag{27}$$

This was found to be effective, especially when $u < \lambda$, which is the case for small values of a. When u exceeded λ, the complementary series was used, namely

$$e^{-(\lambda+u)} \sum_{\nu=0}^{\infty} \left(\frac{u}{\lambda}\right)^{(a-1-\nu)/2} I_{a-1-\nu}(2\sqrt{u\lambda}) = \beta. \tag{28}$$

The calculations proceeded in much the same manner as in the finite case. The values of $2u$ were obtained from Miss Thomson's table of percentage points of $\chi^2_\nu(\alpha)$ distribution with $\nu = 2a$ degrees of freedom, while the values of I_0 and I_1 were obtained from the tables of Bessel Functions [12] for $2\sqrt{\lambda}z \leq 20$, and from the tables of Anding [13] for larger values of the variable. The values of I_ν for $\nu > 1$, were computed from the recurrence relation[4]

[4] If a is an odd integer, the values of $I_\nu(z)$ can be built up using (29) from $I_{-\frac{1}{2}}(z) = \dfrac{\cosh z}{\sqrt{\pi z}}$ and $I_{\frac{1}{2}}(z) = \dfrac{\sinh z}{\sqrt{\pi z}}$, which of course are tabled.

$$I_{\nu+1} = -\frac{2\nu}{z} I_\nu(z) + I_{\nu-1}(z). \tag{29}$$

Future calculations of this sort could make use of the forthcoming tables [14] of $I_\nu(z)$ for $\nu \leq 20$, and $z \leq 20$ to ten significant places. As before, the tabulated values were obtained by correcting trial values of φ by means of the derivative

$$\frac{d\beta}{d\varphi} = \sqrt{2u(2a+1)}e^{-(\lambda+u)}\left(\frac{u}{\lambda}\right)^{(a-1)/2} I_a(2\sqrt{u\lambda}), \tag{30}$$

which again was obtained as a by-product of the calculations. This method becomes impractical when a is too large, because a great deal of accuracy is lost in applying recurrence (29) many times. For some of the larger values of a it was found preferable to use the series

$$1 - \beta = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \frac{\Gamma_u(a+i)}{\Gamma(a+i)} \tag{31}$$

although it converges rather slowly. In other words the upper limits for a and b were pushed as far as was practicable.

The values which are tabled to two instead of three decimals were interpolate using second differences, all other values were computed in the manner described above. Difference tables were made by rows, columns and between tables, as a final check on the work. Difference tables using harmonic interpolation were also made for both rows and columns, and found very effective, with the exception of the lower right hand corner, where φ drops rapidly to zero. The last column of each table is to be used for harmonic interpolation.

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SOME COMBINATORIAL FORMULAS WITH APPLICATIONS TO PROBABLE VALUES OF A POLYNOMIAL-PRODUCT AND TO DIFFERENCES OF ZERO

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1. The main purpose of the paper is to establish some combinatorial formulas concerning the mathematical expectations or probable values of a product of n given polynomials. The problem may be stated more definitely as follows:

Let $x_1, \cdots, x_n$ be n non-negative discontinuous variables for which we have assumed that the probability that each x takes a possible value is equally likely, and let $f_1(x), \cdots, f_n(x)$ be n given polynomials. Then we shall ask: What is the probable value of the product $f_1(x_1) \cdots f_n(x_n)$, provided the sum of the variables $x_1, \cdots, x_n$ is known? More generally, we may consider the problem with certain restrictions to x such as $a \leq x_i \leq b$, $(i = 1 \cdots n)$.

By a limiting process[1] it will be found that all the formulas established for the preceding problem can be extended to the case of continuous variables. On this account, it is important to find explicit formulas for the problem merely involving discontinuous variables.

By the definition[2] of MacMahon, we say that a set of numbers $(x_1 \cdots x_n)$ is over all different compositions of m into n parts with each $x \geq k$, if $(x_1 \cdots x_n)$ runs over all different integer solutions of the linear equation $x_1 + \cdots x_n = m$ with each $x \geq k$. We shall use the notation $(m; k; x_1 \cdots x_n)$, or simply $(m; k; x)$, to denote that a set of numbers $(x_1 \cdots x_n)$ is over all different compositions of m into n parts with each $x \geq k$.

The notation $E(m; \delta; [f_1(x)] \cdots [f_n(x)])$ will be used to denote the mathematical expectation of the product $f_1(x_1) \cdots f_n(x_n)$ in which the sum of n variable quantities $x_1, \cdots, x_n$ is known, namely $x_1 + \cdots + x_n = m$, and each quantity is a multiple of δ and m is of course a multiple of δ. Thus by the definition[3] of mathematical expectations we have

$$(1) \qquad E(m; \delta; [f_1] \cdots [f_n]) = \Big(\sum_{(m/\delta;\, 1;\, x)} 1\Big)^{-1} \sum_{(m/\delta;\, 1;\, x)} f_1(x_1\delta) \cdots f_n(x_n\delta),$$

where the summation on the right-hand side runs over all different compositions of m/δ into n parts with each $x \geq 1$, and the given constant δ is called a "varying unit", that is the least possible difference between two unequal quantities in $(x_1 \cdots x_n)$. If the varying unit approaches zero, $(x_1 \cdots x_n)$ will become a set of continuous variables.

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[1] The limiting process will be illustrated by the proof of corollary 2 of theorem 1 in this paper.

[2] MacMahon, *Combinatory Analysis*, Vol. 1, p. 150.

[3] See for example W. Burnside, *Theory of Probability*, Chap. 4, 13.

If $f_1(x) = \cdots = f_n(x)$, we may write

$$E(m; \delta; [f]^n) \quad \text{instead of} \quad E(m; \delta; [f_1] \cdots [f_n]).$$

A well-known convention for $\binom{m}{n}$ is also adopted here:

$$\binom{m}{n} = \begin{cases} \dfrac{m!}{n!\,(m-n)!}, & \text{if } 0 \le n \le m, \\ 0 & \text{otherwise.} \end{cases}$$

2. Lemmas. In order to obtain explicit formulas desired we first establish four lemmas as follows:

LEMMA 1. *Let $m, r_1, \cdots, r_n$ be non-negative integers. Then*

$$\sum_{(m;\,0;\,x)} \prod_{\nu=1}^{n} \binom{x_\nu}{r_\nu} = \binom{m+n-1}{r_1 + \cdots + r_n + n - 1}. \tag{2}$$

PROOF. Construct a generating function:

$$\left(\frac{1}{1-x}\right)^{r_1+1} \cdots \left(\frac{1}{1-x}\right)^{r_n+1}, \qquad |x| < 1.$$

It is observed that the coefficient of the term $x^{m-(r_1+\cdots+r_n)}$ in the expansion of the above product is given by

$$\sum_{(m-r_1-\cdots-r_n;\,0;\,x)} \binom{r_1+x_1}{x_1} \cdots \binom{r_n+x_n}{x_n} = \sum_{(m;\,0;\,x)} \binom{x_1}{r_1} \cdots \binom{x_n}{r_n}.$$

On the other hand we see that the coefficient of the term $x^{m-(r_1+\cdots+r_n)}$ in the expansion of $\left(\dfrac{1}{1-x}\right)^{r_1+\cdots+r_n+n}$ is given by

$$\binom{r_1 + \cdots + r_n + n + m - (r_1 + \cdots + r_n) - 1}{m - (r_1 + \cdots + r_n)} = \binom{m+n-1}{\Sigma r_1 + n - 1}.$$

Hence the lemma.

LEMMA 2. *Let $a, b, c, \cdots$ be arbitrary constants, and $k_1, k_2, k_3, \cdots$ be positive integers. Then*

$$\sum_{(m;1;x)} \prod_{\nu=1}^{n} \left\{ a\binom{x_\nu}{k_1} + b\binom{x_\nu}{k_2} + c\binom{x_\nu}{k_3} + \cdots \right\}$$
$$= n! \sum_{(n;0;\alpha\beta\gamma\cdots)} \binom{m+n-1}{\alpha k_1 + \beta k_2 + \gamma k_3 + \cdots + n - 1} \frac{a^\alpha}{\alpha!} \frac{b^\beta}{\beta!} \frac{c^\gamma}{\gamma!} \cdots. \tag{3}$$

PROOF. Expanding the left-hand side of (3), we see that the coefficient of the term $a^\alpha b^\beta c^\gamma \cdots$ is equal to

$$\frac{n!}{\alpha!\beta!\gamma!\cdots} \sum_{(m;0;x)} \binom{x_1}{k_1} \cdots \binom{x_\alpha}{k_1} \binom{x_{\alpha+1}}{k_2} \cdots \binom{x_{\alpha+\beta}}{k_2} \binom{x_{\alpha+\beta+1}}{k_3} \cdots \binom{x_{\alpha+\beta+\gamma}}{k_3} \cdots.$$

By lemma 1 it is reduced to

$$\frac{n!}{\alpha!\beta!\gamma!\cdots}\binom{m+n-1}{\alpha k_1+\beta k_2+\gamma k_3+\cdots+n-1}.$$

Substituting, we get the lemma.

From now on, we shall frequently write $f^{(x)}$ instead of $f(x)$, so that

$$f^{(\nu)}g^{(0)}+\binom{\nu}{1}f^{(\nu-1)}g^{(1)}+\cdots+\binom{\nu}{\nu}f^{(0)}g^{(\nu)}=(f+g)^{(\nu)}.$$

LEMMA 3. *Let m, n ($\leq m$) be two positive integers. Then for any given polynomial $f(x)$ of the kth degree we have*

$$(4)\qquad \sum_{(m;1;x)} f(x_1)\cdots f(x_n)=n!\sum_{(n;0;p)}\binom{m+n-1}{S(p)+n-1}\prod_{\nu=0}^{k}\frac{[(f-1)^{(\nu)}]^{p_\nu}}{p_\nu!},$$

where

$$f^{(x)}=f(x),\qquad S(p)=1\cdot p_1+\cdots+k\cdot p_k.$$

PROOF. Since $f(x)$ is a polynomial of the kth degree, there exist $(k+1)$ values $\beta_k,\cdots,\beta_0$ such that

$$\beta_k\binom{x}{k}+\cdots+\beta_1\binom{x}{1}+\beta_0=f(x).$$

By putting $x=0,1,\cdots,k$, we find successively[4]

$$(5)\qquad \beta_\nu=f^{(\nu)}-\binom{\nu}{1}f^{(\nu-1)}+\cdots+(-1)^\nu\binom{\nu}{\nu}f^{(0)}=(f-1)^{(\nu)},$$

$$(\nu=0,1,\cdots,k).$$

The lemma is thus obtained by means of (3).

For convenience, we denote the summation

$$\sum_{(m;1;x)} f_1(x_1)\cdots f_n(x_n)\quad\text{by}\quad S(m,[f_1]\cdots[f_n]).$$

Thus formula (4) can be rewritten:

$$(4)'\qquad S(m,[f]^n)=n!\sum_{(n;0;p)}\binom{m+n-1}{1\cdot p_1+\cdots+k\cdot p_k+n-1}\frac{\beta_0^{p_0}}{p_0!}\frac{\beta_1^{p_1}}{p_1!}\cdots\frac{\beta_k^{p_k}}{p_k!},$$

where $\beta_0,\beta_1,\cdots,\beta_k$ are given by (5).

LEMMA 4. *Let $f_1(x),\cdots,f_n(x)$ be n given polynomials, Then*

$$(6)\qquad S(m,[f_1]\cdots[f_n])=\frac{1}{n!}\sum_{(\nu_1\cdots\nu_k)\in(1\cdots n)}(-1)^{n-k}S(m,[f_{\nu_1}+\cdots+f_{\nu_k}]^n),$$

[4] Strictly speaking, the relation (5) is established and proved by induction on ν, in which a well known combinatorial equality is applied.

where the summation on the right-hand side runs over all different combinations out of $(1 \cdots n)$, $(k = 1 \cdots n)$.

For example, if $n = 3$, then all the different combinations out of (123) will be: (1), (2), (3), (12), (13), (23), (123).

PROOF. Consider a typical term

$$\frac{n!}{q_1! \cdots q_t!} S(m, [f_{\nu_1}]^{q_1} \cdots [f_{\nu_t}]^{q_t}),$$

where $1 \leq t \leq n$, $q_1 + \cdots + q_t = n$. Now a necessary and sufficient condition that the term will be contained in the expansion of $S(m, [f_{\mu_1} + \cdots + f_{\mu_k}]^n)$ is $(\nu_1 \cdots \nu_t)\epsilon(\mu_1 \cdots \mu_k)$, i.e. $(\nu_1 \cdots \nu_t)$ is a combination out of $(\mu_1 \cdots \mu_k)$, and for a fixed k, there are $\binom{n-t}{k-t}$-different combinations of $(\mu_1 \cdots \mu_k)$ satisfying the condition $(\nu_1 \cdots \nu_t)\epsilon(\mu_1 \cdots \mu_k)\epsilon(1 \cdots n)$. Therefore the number of occurrences of the term in the right-hand side of (6) is given by

$$\sum_{\nu=0}^{n-t} (-1)^\nu \binom{n-t}{\nu} = (1-1)^{n-t} = \begin{cases} 0 & \text{if } t < n, \\ 1 & \text{if } t = n. \end{cases}$$

The term vanishes generally except $q_1 = \cdots = q_t = 1$. Hence the right-hand side gives $S(m, [f_1] \cdots [f_n])$.

3. Theorems and corollaries. In the following statement of theorems and corollaries the notation $(x_1 \cdots x_n)$ will be always used to denote a set of undetermined quantities as specified.

THEOREM 1. *Suppose that* $(x_1 \cdots x_n)$ *is a set of natural numbers in which only the sum of the numbers is known, namely* $x_1 + \cdots + x_n = m$. *Then, for any given polynomial* $f(x)$ *of the* k*th degree, the mathematical expectation of* $f(x_1) \cdots f(x_n)$ *is given by*

$$E(m, 1, [f]^n) = n! \binom{m-1}{n-1}^{-1} \sum_{(n;0;p)} \binom{m+n-1}{S(p)+n-1} \frac{[(f-1)^{(0)}]^{p_0} \cdots [(f-1)^{(k)}]^{p_k}}{p_0! \cdots p_k!}. \tag{7}$$

PROOF. Let $m' = m + nr$. By lemma 1 we have

$$\sum_{(m;0;x)} \binom{x_1}{0} \cdots \binom{x_n}{0} = \sum_{(m;0;x)} 1 = \sum_{(m';r;x)} 1 = \binom{m' - nr + n - 1}{n-1}.$$

This is the number of compositions of m' into n parts with each part $\geq r$. In particular, if $r = 1$, we find that the number of compositions of m into n parts is $\binom{m-1}{n-1}$. Thus by (1), the required value is equal to

$$\{S(m, [1]^n)\}^{-1} S(m, [f]^n), \quad \text{i.e.} \quad \binom{m-1}{n-1}^{-1} S(m, [f]^n).$$

The theorem is therefore proved by lemma 3.

COROLLARY 1. *Let $(x_1 \cdots x_n)$ be a set of positive quantities, of which the varying unit is δ, and the sum is m. Then, for any given polynomial $f(x)$ of the kth degree, we have*

$$E(m, \delta, [f]^n)$$

$$= n!\left(\binom{\frac{m}{\delta}}{n-1}\right)^{-1} \sum_{(n;0;p)} \binom{\frac{m}{\delta}+n-1}{S(p)+n-1} \frac{[(g-1)^{(0)}]^{p_0} \cdots [(g-1)^{(k)}]^{p_k}}{p_0! \cdots p_k!}, \tag{8}$$

where

$$g(x) = f(x\delta), \qquad S(p) = p_1 + \cdots + kp_k .$$

PROOF. It follows immediately by the relation:

$$E(m, \delta, [f(x)]^n) = E\left(\binom{\frac{m}{\delta}}{}, 1, [f(\delta x)]^n\right).$$

COROLLARY 2. *Let $(x_1 \cdots x_n)$ be a set of non-negatively real numbers, of which the sum is known, namely $x_1 + \cdots + x_n = m$, m being a known real number. Then, for any given polynomial $f(x) = a_0 + \cdots + a_k x^k$, $(a_k \neq 0)$ we have*

$$E(m, 0, [f]^n)$$

$$= \frac{(n!)^2}{n} \sum_{(n;0;p)} \frac{m^{1\cdot q_1 + \cdots + k\cdot q_k}}{(1\cdot q_1 + \cdots + kq_k + n - 1)!} \frac{(0!a_0)^{q_0}}{q_0!} \cdots \frac{(k!a_k)^{q_k}}{q_k!}. \tag{9}$$

PROOF. The proof of the corollary depends essentially on the concept that two unequal real numbers may differ by an arbitrary small number h.

Let h be an arbitrary positive number and write $f(xh)/h^k = g(x, h)$, where the number k is the degree of $f(x)$. Then, since

$$\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu}(n-\nu)^p = \begin{cases} 0 & \text{if } p < n \\ n! & \text{if } p = n \\ \binom{n+1}{2} n! & \text{if } p = n+1, \end{cases}$$

we may write

$$g(\nu, h) - \binom{\nu}{1} g(\nu - 1, h) + \cdots + (-1)^\nu \binom{\nu}{\nu} g(0, h) = h^{\nu-k}\{\nu! a_\nu + h \cdot R_\nu(h)\},$$

$$\nu = 0, 1, \cdots k,$$

where

$$\lim_{h\to 0} R_\nu(h) = \left\{\binom{\nu}{0}\nu^{\nu+1} - \binom{\nu}{1}(\nu-1)^{\nu+1} + \cdots + (-1)^{\nu+1}\binom{\nu}{\nu-1}\right\} a_{\nu+1}$$

$$= \binom{\nu+1}{2} \nu! a_{\nu+1} .$$

Now we pass to the limit $h \to 0$ in which we assume that h runs through a sequence of real numbers of the form $\frac{m}{N}$, N being a natural number. Thus by corollary 1 we have

$$\lim_{h\to 0} E(m, h, [f]^n)$$

$$= n! \lim_{h\to 0} \sum_{(n;0;p)} \binom{\binom{m}{h} + n - 1}{S(p) + n - 1} \binom{\binom{m}{h} - 1}{n - 1}^{-1} h^{nk} \prod_{\nu=0}^{k} \frac{\{\nu! a_\nu + h \cdot R_\nu(h)\}^{p_\nu}}{h^{(k-\nu)p_\nu} \cdot p_\nu!},$$

$$= n! \sum_{(n;0;p)} \left\{ \lim_{h\to 0} \frac{\binom{m}{h} + n - 1)!(n-1)! \binom{\binom{m}{h} - n}{} ! h^{S(p)}}{(S(p) + n - 1)! \binom{\binom{m}{h} - S(p)}{} ! \binom{\binom{m}{h} - 1}{} !} \right\}$$

$$\left\{ \lim_{h\to 0} \prod_{\nu=0}^{k} \frac{[\nu! a_\nu + h \cdot R_\nu(h)]^{p_\nu}}{p_\nu!} \right\},$$

$$= \sum_{(n;0;p)} \frac{n!(n-1)! m^{S(p)}}{(S(p) + n - 1)!} \prod_{\nu=0}^{k} \frac{(\nu! a_\nu)^{p_\nu}}{p_\nu!}, \qquad (S(p) = 1 \cdot p_1 + \cdots + k \cdot p_k).$$

Hence the corollary.[5]

COROLLARY 3. *Let $(x_1 \cdots x_n)$ be a set of positive real numbers under a known condition $a \leq x_1 + \cdots + x_n \leq b$, where a $(< b)$, b are two positive real numbers. Then, for any given polynomial $f(x) = a_0 + \cdots + a_k x^k$, $(a_k \neq 0)$, the mathematical expectation of the product $f(x_1) \cdots f(x_n)$ which we denote by $E((ab), 0, [f]^n)$ is given by the formula*

$$(10) \quad E((ab), 0, [f]^n) = \frac{n!(n-1)!}{b-a} \sum_{(n;0;q)} \frac{b^{1+S(q)} - a^{1+S(q)}}{(1 + S(q)) \cdot (n - 1 + S(q))!} \cdot \frac{a_0^{q_0} \cdots (k!\, a_k)^{q_k}}{q_0! \cdots q_k!}, \quad (S(q) = q_1 + \cdots + kq_k).$$

PROOF. Since the probability that the sum of $x_1, \cdots, x_n$ takes a value between a and b is equally likely, we see that the required mathematical expectation will be the mean of $\int_a^b E(u, 0, [f]^n) du$, that is

$$E((ab), 0, [f]^n) = \frac{1}{b-a} \int_a^b E(u, 0, [f]^n)\, du.$$

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[5] This corollary can also be proved by means of Dirichlet's integral. In fact, the right-hand side of (9) is given by the quotient of the two integrals:

$$(\int \cdots \int f(x_1) \cdots f(x_n)\, dx_1 \cdots dx_n) / (\int \cdots \int dx_1 \cdots dx_n),$$

the integrals being taken over the region: $x_1 + \ldots + x_n = m, x_1 \geq 0, \ldots, x_n \geq 0$.

The formula (10) is therefore obtained by integrating the right-hand side of (9) and dividing it by $(b - a)$.

On the other hand we find that

$$\lim_{h\to 0} E((a, a + h), 0, [f]^n) = E(a, 0, [f]^n).$$

This shows that the corollary 2 can be also deduced from 3.

THEOREM 2. (A generalization of theorem 1). *Let $f_1(x), \cdots, f_n(x)$ be n given polynomials whose degrees do not exceed k. Then we have*

$$E(m, 1, [f_1] \cdots [f_n])$$

$$= \sum_{(\nu_1\cdots\nu_t)\,\epsilon\,(1\cdots n)} \sum_{(n;0;p)} (-1)^{n-t} \frac{\binom{m+n-1}{S(p)+n-1}}{\binom{m-1}{n-1}} \prod_{j=0}^{k} \frac{[(f_{\nu_1\cdots\nu_t} - 1)^{(j)}]^{p_j}}{p_j!}, \tag{11}$$

where

$$f_{\nu_1\cdots\nu_t} = f_{\nu_1}(x) + \cdots + f_{\nu_t}(x), \qquad S(p) = 1\cdot p_1 + \cdots + k\cdot p_j .$$

PROOF. In the proof of theorem 1 we have shown that

$$E(m, 1, [f]^n) = \binom{m-1}{n-1}^{-1} S(m, [f]^n).$$

Hence, by similar reasoning and lemma 4 we obtain

$$\begin{aligned} E(m, 1, [f_1] \cdots [f_n]) &= \binom{m-1}{n-1}^{-1} S(m, [f_1] \cdots [f_n]) \\ &= \sum_{(\nu_1\cdots\nu_t)\,\epsilon\,(1\cdots n)} \frac{(-1)^{n-t}}{n!\binom{m-1}{n-1}} S(m, [f_{\nu_1} + \cdots + f_{\nu_t}]^n) \\ &= \sum_{(\nu_1\cdots\nu_t)\,\epsilon\,(1\cdots n)} \sum_{(n;0;p)} (-1)^{n-t} \frac{\binom{m+n-1}{S(p)+n-1}}{\binom{m-1}{n-1}} \\ &\qquad \prod_{j=0}^{k} \frac{[(f_{\nu_1\cdots\nu_t} - 1)^{(j)}]^{p_j}}{p_j!}. \end{aligned}$$

Theorem 2 is proved.

COROLLARY 1. *Let δ be a varying unit. Then*

$$E(m, \delta, [f_1] \cdots [f_n]) = \sum_{(\nu_1\cdots\nu_t)\,\epsilon\,(1\cdots n)} \sum_{(n;0;p)} (-1)^{n-t} \binom{\left(\frac{m}{\delta}\right)+n-1}{S(p)+n-1} \cdot \binom{m-1}{n-1}^{-1} \prod_{j=0}^{k} \frac{[(g_{\nu_1\cdots\nu_t} - 1)^{(j)}]^{p_j}}{p_j!}, \tag{12}$$

where

$$g_\nu(x) = f_\nu(x\delta), (\nu = 1 \cdots n) \quad \text{and} \quad g_{\nu_1 \cdots \nu_t} = g_{\nu_1}(x) + \cdots + g_{\nu_t}(x).$$

PROOF. This is almost trivial because of

$$E(m,\ \delta,\ [f_1(x)] \cdots [f_n(x)]) = E\left(\left(\frac{m}{\delta}\right),\ 1,\ [f_1(x\delta)] \cdots [f_n(x\delta)]\right).$$

COROLLARY 2.[6] *For any given positive real number m, we have*

$$E(m, 0, [x^{p_1}] \cdots [x^{p_n}]) = \frac{p_1! \cdots p_n!\,(n-1)!}{(p_1 + \cdots + p_n + n - 1)!}\, m^{p_1 + \cdots + p_n}. \tag{13}$$

PROOF. By a passage to the limit $\delta \to 0$, we get

$$\lim_{\delta \to 0} E(m, \delta, [f_1] \cdots [f_n]) = \lim_{\delta \to 0} \sum \frac{(-1)^{n-t}}{n!} E(m, \delta, [f_{\nu_1} + \cdots + f_{\nu_t}]^n)$$

$$= \sum \frac{(-1)^{n-t}}{n!} \lim_{\delta \to 0} E(m, \delta, [f_{\nu_1 \cdots \nu_t}]^n).$$

i.e.

$$E(m, 0, [f_1] \cdots [f_n]) = \sum \frac{(-1)^{n-t}}{n!} E(m, 0, [f_{\nu_1 \cdots \nu_t}]^n).$$

The corollary is then deduced by (9).

THEOREM 3. (Further generalization of theorem 1). *Let $(x_1 \cdots x_n)$ be a set of arbitrary integers restricted to the conditions:*

$$x_1 + \cdots + x_n = m, \qquad a \le x_i \le b,$$

where m, a, b are all known integers. Then for any given polynomial $f(x)$, the mathematical expectation of the product $f(x_1) \cdots f(x_n)$ denoted by $\underset{(a,b)}{E}(m, 1, [f]^n)$ is given by the following

$$\underset{(a,b)}{E}(m, 1, [f]^n) = \frac{\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} S(m', [g]^\nu [h]^{n-\nu})}{\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu}\binom{m'-1}{n-1}}, \tag{14}$$

where

$$g(x) = f(x + b), \quad h(x) = f(x + a - 1), \quad m' = m - (a-1)n + (a - b - 1)\nu.$$

PROOF. Define

$$S(m, [f]^n) = 0 \quad \text{for } m < n \text{ and } \quad S(m, [f]^0) = \begin{cases} 0 & \text{for } m > 0, \\ 1 & \text{for } m = 0. \end{cases}$$

[6] This can also be deduced by Dirichlet's integrals.

We shall show that

$$\sum_{\nu=0}^{n} (-1)^{\nu} \binom{n}{\nu} S(m', [g]^{\nu}[h]^{n-\nu}) = \sum_{\substack{(x_1 \cdots x_n) \\ a \le x \le b}} f(x_1) \cdots f(x_n),$$

where on the right-hand side of the expression the set $(x_1 \cdots x_n)$ under the summation runs over all different compositions of m into n parts with each $a \le x_i \le b$, $(i = 1 \cdots n)$.

We denote the left-hand side of the expression by $\mathfrak{S}$, then by decomposing $S(m', [g]^{\nu}[h]^{n-\nu})$ we have

$$\mathfrak{S} = \sum_{\nu=0}^{n} \sum_{\bar{m}=\nu}^{m'} (-1)^{\nu} \binom{n}{\nu} S(\bar{m}, [f(x+b)]^{\nu}) S(m' - \bar{m}, [f(x+a-1)]^{n-\nu}).$$

Let $f(\bar{x}_1) \cdots f(\bar{x}_n)$ be a term contained in $\mathfrak{S}$, i.e. $\bar{x}_1 + \cdots + \bar{x}_n = m$, $\bar{x}_1 \ge a, \cdots, \bar{x}_n \ge a$. And we suppose that $\bar{x}_{\nu_1} \ge b+1, \cdots, \bar{x}_{\nu_t} \ge b+1$, where $\nu_i \ne \nu_j$ if $i \ne j$. Then it is found that the number of occurrences of the term in $\mathfrak{S}$ is given by

$$\sum_{s=0}^{t} (-1)^{s} \binom{t}{s} = (1-1)^{t} = \begin{cases} 0 & \text{if} \quad t \ge 1, \\ 1 & \text{if} \quad t = 0. \end{cases}$$

This shows that the term $f(\bar{x}_1) \cdots f(\bar{x}_n)$ of $\mathfrak{S}$ generally vanishes except $a \le x_i \le b$. Hence we have

$$\mathfrak{S} = \sum_{\substack{(x_1 \cdots x_n) \\ a \le x \le b}} f(x_1) \cdots f(x_n).$$

Next, we shall find the number of compositions of m into n parts with each $a \le x_i \le b$, i.e. the number of terms in $\mathfrak{S}$. By the result just obtained we see that the number is given by

$$\sum_{\nu=0}^{n} \sum_{\bar{m}=0}^{m'} (-1)^{\nu} \binom{n}{\nu} \sum_{(\bar{m};1;x)} 1 \sum_{(m'-\bar{m};1;x)} 1$$

$$= \sum_{\nu=0}^{n} (-1)^{\nu} \binom{n}{\nu} \left\{ \sum_{\bar{m}=0}^{m'} \binom{\bar{m}-1}{\nu-1} \binom{m'-\bar{m}-1}{n-\nu-1} \right\}$$

$$= \sum_{\nu=0}^{n} (-1)^{\nu} \binom{n}{\nu} \binom{m'-1}{n-1}.$$

Hence the theorem.

The theorem just proved shows that the mathematical expectation $E_{(a,b)}(m, 1, [f]^n)$ can be expressed by $S(\bar{m}, [g]^{\nu})$ and is therefore expressible[7] in terms of the linear combinations of the coefficients of the polynomial $f(x)$.

COROLLARY 1. *Let δ be a varying unit for which $\frac{m}{\delta}, \frac{a}{\delta}, \frac{b}{\delta}$ are all integers. Then*

$$E_{(a,b)}(m, \delta, [f(x)]^n) = E_{\left(\frac{a}{\delta} \frac{b}{\delta}\right)}\left(\frac{m}{\delta}, 1, [f(\delta x)]^n\right).$$

[7] See lemmas 3 and 4.

COROLLARY 2. *Let $f_1(x), \cdots, f_n(x)$ be n given polynomials. Then*

$$\underset{(a,b)}{E}(m, \delta, [f_1] \cdots [f_n]) = \sum_{(\nu_1 \cdots \nu_t) \in (1 \cdots n)} \frac{(-1)^{n-t}}{n!} \underset{(a,b)}{E}(m, \delta, [f_{\nu_1} + \cdots + f_{\nu_t}]^n).$$

COROLLARY 3. *The number of compositions of m into n parts with each $a_i \leq x_i \leq b_i$, $(i = 1 \cdots n)$ is equal to*

$$\sum_{\nu_1=0,\cdots,\nu_n=0}^{1,\cdots,1} (-1)^{\nu_1+\cdots+\nu_n} \cdot \binom{m+n-(a_1+\cdots+a_n)+(a_1-b_1-1)\nu_1+\cdots+(a_n-b_n-1)\nu_n-1}{n-1}$$

PROOF. We have shown that the number of compositions of m into n parts with each $a \leq x \leq b$ is given by

$$\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} \binom{m-(a-1)n+(a-b-1)\nu-1}{n-1}.$$

Hence the number of integer solutions of the equation

$$x_{11} + \cdots + k_{1n_1} + \cdots + x_{s1} + \cdots + x_{sn_s} = m$$

with each $a_\nu \leq x_{\nu\mu} \leq b_\nu (\nu = 1 \cdots s; \mu = 1 \cdots n_\nu)$ is given by

$$\sum_{(m;1;m_1 \cdots m_s)} \sum_{\nu_1=0}^{n_1} \cdots \sum_{\nu_s=0}^{n_s} (-1)^{\nu_1+\cdots+\nu_s} \prod_{i=1}^{s} \binom{n_i}{\nu_i} \cdot \binom{m_i-(a_i-1)n_i+(a_i-b_i-1)\nu_i-1}{n_i-1}$$

$$= \sum_{\nu_1=0}^{n_1} \cdots \sum_{\nu_s=0}^{n_s} (-1)^{\nu_1+\cdots+\nu_s} \prod_{i=1}^{s} \binom{n_i}{\nu_i} \cdot \left\{ \sum_{(m;1;m_i)} \prod_{i=1}^{s} \binom{m_i-(a_i-1)n_i+(a_i-b_i-1)\nu_i-1}{n_i-1} \right\}$$

$$= \sum_{\nu_1=0,\cdots,\nu_s=0}^{1,\cdots,1} (-1)^{\nu_1+\cdots+\nu_s} \binom{n_1}{\nu_1} \cdots \binom{n_s}{\nu_s} \cdot \binom{m-1-\sum(a_1-1)n_1+\sum(a_1-b_1-1)\nu_1}{n_1+\cdots+n_s-1}.$$

The corollary follows at once by putting $n_1 = \cdots = n_s = 1, s = n$.

The last corollary may be restated as follows: Let there be n stores, $b_1, \cdots, b_n$ being the numbers of stocks contained in 1st, 2nd, $\cdots$, nth store respectively. Then m stocks containing at least a_i stocks of the ith store can be taken from these stores in

$$\sum_{\nu_1=0,\cdots,\nu_n=0}^{1,\cdots,1} (-1)^{\nu_1+\cdots+\nu_n} \binom{m+n-1-\sum a_1+\sum(a_1-b_1-1)\nu_1}{n-1}$$

— different ways.

We have now established several combinatorial formulas concerning the mathematical expectations of the product $f_1(x_1) \cdots f_n(x_n)$ under certain conditions. Apparently, there are many examples which can be solved by means of the results just obtained. For brevity, we may state a general criterion as follows: The mathematical expectation of a function, $F(x_1, \cdots, x_n)$ say, can be estimated by the above mentioned formulas, if and only if 1) the sum of $x_1, \cdots, x_n$ is known, and 2) there exist n polynomials $f_1(x), \cdots, f_n(x)$ such that $F(x_1, \cdots x_n)$ is proportional to $f_i(x_i)$, $(i = 1 \cdots n)$. The undetermined quantities in $(x_1, \cdots, x_n)$ may or may not be continuous, if the quantities are discontinuous, the varying unit is necessarily known.

4. Convenient formulas for differences of zero.[8]

Given $f(x) = \beta_0 + \beta_1 x + \cdots + \beta_k x^k (\beta_k \neq 0)$ we may write

$$(f-1)^{(\nu)} = \sum_{s=0}^{k} \nu! \beta_s S_{\nu,s} = \sum_{s=0}^{k} \beta_s \Delta^\nu 0^s,$$

where $S_{\nu,s}$ is a Stirling number of the second kind, as used by Jordan and defined by

$$\nu! S_{\nu,s} = \Delta^\nu 0^s = \sum_{x=0}^{\nu} (-1)^{\nu-x} \binom{\nu}{x} x^s,$$

$\Delta^\nu 0^s$ being in the language of the calculus of finite differences, "a difference of zero".

In terms of the differences of zero, the formulas (7) and (11) may also be restated as follows:

$$E(m, 1, [f]^n) = \sum_{(n;\, 0;\, p)} \frac{(m+n-1)!(m-n)!n!(n-1)!}{(m-S(p))!(S(p)+n-1)!(m-1)!} \times \prod_{\nu=0}^{k} \frac{1}{p_\nu!} (\beta_\nu \Delta^\nu 0^\nu + \cdots + \beta_k \Delta^\nu 0^k)^{p_\nu}. \tag{7'}$$

$$E(m, 1, [f_1] \cdots [f_n]) = \sum_{(p_1 \cdots p_t) \in (1 \cdots n)} (-1)^{n-t} \sum_{(n;\, 0;\, p)} \frac{(m+n-1)!(m-n)!(n-1)!}{(m-S(p))!(S(p)+n-1)!(m-1)!} \times \prod_{\nu=0}^{k} \frac{1}{p_\nu!} (B_\nu \Delta^\nu 0^\nu + \cdots + B_k \Delta^\nu 0^k)^{p_\nu}, \tag{11'}$$

where

$$f_\nu(x) = \beta_{\nu 0} + \cdots + \beta_{\nu k} x^k, \quad B_\mu = \beta_{1\mu} + \cdots + \beta_{n\mu}, \quad S(p) = 1 \cdot p_1 + \cdots + k p_k .$$

[8] The methods for obtaining convenient formulas for differences of zero as stated in the first part of this paragraph are similar to those used by Paul S. Dwyer in his paper "The computation of moments with cumulative totals," *Annals of Math. Stat.*, Vol. 9 (1938), pp. 288–303.

The formulas (7)′ and (11)′ tell us that the difference of zero plays an important rôle in the calculation of mathematical expectations of a polynomial product under known conditions. On account of this fact, we are now going to investigate some recurrence relations and approximations for the differences of zero.

As m is larger than t, we may find a convenient recurrence relation as follows:

$$\frac{\Delta^m 0^{m+t}}{m!} = S_{m,m+t} = \lambda_1(t)\binom{m+t}{t+1} + \lambda_2(t)\binom{m+t}{t+2} + \cdots + \lambda_t(t)\binom{m+t}{t+t},$$

(15)

$$\frac{\Delta^m 0^{m+t+1}}{m!} = S_{m,m+t+1} = \lambda_1(t+1)\binom{m+t+1}{t+1+1} + \lambda_2(t+1)\binom{m+t+1}{t+1+2} + \cdots + \lambda_{t+1}(t+1)\binom{m+t+1}{2t+2},$$

where

$$(t+j)\lambda_{j-1}(t) + j\cdot\lambda_j(t) \equiv \lambda_j(t+1), \quad \lambda_0 \equiv \lambda_{t+1}(t) \equiv 0, \quad \lambda_1(t) \equiv \lambda_1(t+1) \equiv 1,$$

and $\lambda_2(t), \cdots, \lambda_t(t)$ are all independent of m but depending on t.

Starting with the first equation of (15) and using a well known relation (due to Jordan) $S_{m,n+1} = S_{m-1,n} + m\cdot S_{m,n}$, successively, we get

$$\begin{aligned}
S_{m,m+t+1} &= \sum_{\nu=1}^{m} (m-\nu+1) S_{m-\nu+1,m+t+1-\nu} \\
&= \sum_{j=1}^{t} \lambda_j(t) \sum_{\nu=1}^{m} \binom{m+t+1-\nu}{t+j} (m-\nu+1) \\
&= \sum_{j=1}^{t} \lambda_j(t) \left\{ \sum_{\nu=1}^{m} \binom{m+t+1-\nu}{t+j+1} (t+j+1) + \sum_{\nu=1}^{m} \binom{m+t+1-\nu}{t+j} \cdot j \right\} \\
&= \sum_{j=1}^{t} \lambda_j(t) \left\{ \binom{m+t+1}{t+j+2} (t+j+1) + \binom{m+t+1}{t+j+1} \cdot j \right\} \\
&= \sum_{j=1}^{t+1} \{(t+j)\lambda_{j-1}(t) + j\lambda_j(t)\} \binom{m+t+1}{t+j+1}.
\end{aligned}$$

The recurrence relation thus follows.

By successive applications of the relation $\lambda_j(t+1) = (t+j)\cdot\lambda_{j-1}(t) + j\cdot\lambda_j(t)$, after nth time say, we may express $\lambda_j(t)$ as a linear combination of $\lambda_j(t-n), \cdots, \lambda_{j-n}(t-n)$, but the coefficients are too complicated.

For $t \leq 9$, by applying the recurrence relation as obtained above, the coefficients may be exhibited as follows:

t	$\lambda_2(t)$	$\lambda_3(t)$	$\lambda_4(t)$	$\lambda_5(t)$	$\lambda_6(t)$	$\lambda_7(t)$	$\lambda_8(t)$	$\lambda_9(t)$
1								
2	3							
3	10	15						
4	25	105	105					
5	56	490	1260	945				
6	119	1918	9450	17325	10395			
7	246	6825	56980	190575	270270	135135		
8	501	22935	302995	1636635	4099095	4729725	2027025	
9	1012	74316	1487200	12122110	47507460	94594500	91891800	34459425

For example, when $t = 4$ we have, according to the table:

$$\Delta^m 0^{m+4} = \left\{\binom{m+4}{5} + 25\binom{m+4}{6} + 105\binom{m+4}{7} + 105\binom{m+4}{8}\right\} m!.$$

We shall now proceed to find some approximations for $S_{n,n+t}$ and $\Delta^n 0^{n+t}$. Firstly, we may write

$$S_{n,n+t} \equiv \frac{\Delta^n 0^{n+t}}{n!} \equiv \lambda_1(t)\binom{n+t}{t+1} + \cdots + \lambda_t(t)\binom{n+t}{2t}.$$

According to the recurrence relation we have

(i) $\lambda_t(t) = (2t-1)\lambda_{t-1}(t-1)$,

(ii) $\lambda_{t-1}(t) = 2(t-1)\cdot\lambda_{t-2}(t-1) + (t-1)\cdot\lambda_{t-1}(t-1)$,

(iii) $\lambda_{t-2}(t) = (2t-3)\cdot\lambda_{t-3}(t-1) + (t-2)\cdot\lambda_{t-2}(t-1)$.

Hence

$$\lambda_t(t) = \frac{(2t)!}{t!\cdot 2^t}; \qquad \lambda_{t-1}(t) = (t-1)!2^{t-1}\sigma(t);$$

$$\lambda_{t-2}(t) = 2^{t-2}\sum_{j=0}^{t-3}(t-2-j)!(t-2-j)\cdot(t-1.5)_j\cdot\sigma(t-1-j),$$

where

$$\sigma(k) = \sum_{x=1}^{k-1}\frac{x}{2^{2x}}\binom{2x}{x}, \qquad (t-1.5)_j = (t-1.5)(t-2.5)\cdots(t-j-0.5).$$

Evidently, the orders of $\binom{n+t}{2t-1}, \binom{n+t}{2t-2}, \cdots, \binom{n+t}{t+1}$ are all less than $2t$ as n tends to infinity.

Now, it can be easily found that

$$\lambda_t(t)\binom{n+t}{2t} = \frac{(2t)!}{t!2^t}\binom{n+t}{2t} = \left(\frac{n^2}{2}\right)^t \frac{1}{t!}\left(1+\frac{t}{n}\right)\cdots\left(1-\frac{t-1}{n}\right)$$

$$= \frac{1}{t!}\left(\frac{n^2}{2}\right)^t \left(1+\frac{t}{n}\right)\left(1-\frac{1}{n^2}\right)\cdots\left(1-\frac{(t-1)^2}{n^2}\right)$$

$$= \frac{1}{t!}\left(\frac{n^2}{2}\right)^t \left\{1+\frac{t}{n}-\frac{(t-1)t(2t-1)}{6n^2}+0\left(\frac{1}{n^3}\right)\right\};$$

$$\lambda_{t-1}(t)\binom{n+t}{2t-1} = \frac{2^t\cdot t!}{n-t+1}\binom{n+t}{2t}\sigma(t) = \frac{2^{2t}t!t!}{(n-t+1)\cdot(2t)!}\sigma(t)\binom{n+t}{2t}\lambda_t(t)$$

$$= \frac{1}{t!}\left(\frac{n^2}{2}\right)^t \left\{\frac{2^{2t}\sigma(t)}{n\binom{2t}{t}}+\frac{(2t-1)\cdot 2^{2t}\sigma(t)}{n^2\binom{2t}{t}}+0\left(\frac{1}{n^3}\right)\right\};$$

$$\lambda_{t-2}(t)\binom{n+t}{2t-2} = \frac{1}{t!}\left(\frac{n^2}{2}\right)^t \left\{\frac{2^t\cdot t!\cdot\lambda_{t-2}(t)}{n^2\cdot(2t-2)!}+0\left(\frac{1}{n^3}\right)\right\}.$$

Hence, we may write

$$S_{n,n+t} = \frac{1}{t!}\left(\frac{n^2}{2}\right)^t \left\{1+\frac{\rho_1}{n}+\frac{\rho_2}{n^2}+0\left(\frac{1}{n^3}\right)\right\}, \tag{16}$$

where

$$\rho_1 = t + 2^{2t}\binom{2t}{t}^{-1}\sigma(t) = t + \frac{4^t\cdot t!t!\sigma(t)}{(2t)!},$$

$$\rho_2 = -\tfrac{1}{6}t(t-1)(2t-1) + \frac{4^t\cdot t!t!(2t-1)}{(2t)!}\sigma(t) + \frac{2^t\cdot t!}{(2t-2)!}\lambda_{t-2}(t).$$

Moreover, it can be shown by Wallis' formula that

$$\sqrt{\frac{x-1}{\pi}} < \frac{x}{2^{2x}}\binom{2x}{x} < \sqrt{\frac{x}{\pi}}, \qquad (x = 1, 2, 3, \cdots).$$

Thus we have

$$\sigma(t) = \sum_{x=1}^{t-1}\frac{x}{2^{2x}}\binom{2x}{x} < \sum_{x=1}^{t-1}\sqrt{\frac{x}{\pi}} < \int_1^t \sqrt{\frac{x}{\pi}}\,dx = \frac{2}{3\sqrt{\pi}}(t^{\frac{3}{2}}-1);$$

$$\sigma(t) = \sum_{x=1}^{t-1}\frac{x}{2^{2x}}\binom{2x}{x} > \sum_{x=0}^{t-2}\sqrt{\frac{x}{\pi}} > \int_0^{t-2}\sqrt{\frac{x}{\pi}}\,dx = \frac{2}{3\sqrt{\pi}}(t-2)^{\frac{3}{2}}.$$

Again, by Wallis' formula we have

$$\sqrt{\pi t} < 4^t\binom{2t}{t}^{-1} < \sqrt{\frac{\pi t^2}{t-1}}.$$

Combining these inequalities we get

$$\tfrac{2}{3}\sqrt{t}\,(t-2)^{\frac{1}{2}} < 4^t\binom{2t}{t}^{-1}\sigma(t) < \frac{2}{3}\sqrt{\frac{t^2}{t-1}}\,(t^{\frac{1}{2}}-1),$$

where

$$\tfrac{2}{3}\sqrt{t}\,(t-2)^{\frac{1}{2}} \sim \tfrac{2}{3}t^2, \qquad \frac{2}{3}\sqrt{\frac{t^2}{t-1}}\,(t^{\frac{1}{2}}-1) \sim \tfrac{2}{3}t^2.$$

Therefore,

$$t + \tfrac{2}{3}\sqrt{t}\sqrt{(t-2)^3} < \rho_1 < t + \tfrac{2}{3}\sqrt{t+1}\,(\sqrt{t^3}-1), \qquad \rho_1 \sim \tfrac{2}{3}t^2.$$

Next, by Stirling's formula

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left\{1 + \frac{1}{12n} + \frac{1}{288n^2} + 0\left(\frac{1}{n^3}\right)\right\}$$

we obtain

$$\Delta^n 0^{n+t} = \left(\frac{n}{e}\right)^n \left(\frac{n^2}{2}\right)^t \frac{\sqrt{2\pi n}}{t!}\left\{1 + \frac{\theta_1}{n} + \frac{\theta_2}{n^2} + 0\left(\frac{1}{n^3}\right)\right\} \sim \frac{n^{n+2t}\sqrt{2\pi n}}{e^n \cdot 2^t \cdot t!}\left(1 + \frac{2t^2}{3n}\right),$$

where

$$\theta_1 = \rho_1 + \tfrac{1}{12} \sim \tfrac{2}{3}t^2,$$

$$\theta_2 = \rho_2 + \tfrac{1}{12}\rho_1 + \tfrac{1}{288}.$$

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RANGES AND MIDRANGES[1]

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1. Introduction. In the following the generating functions of the extremes are studied in order to determine the nature of the distributions of the ranges and the midranges.

A large sample of size n is considered to be drawn from an unlimited symmetrical continuous distribution with zero mean. The difference and the sum of the largest and of the smallest observation, the extremes, are called *range* and *midrange*. R. A. Fisher and L. H. C. Tippett [2] have established the limiting distributions of the largest and of the smallest member of a sample. The exact conditions under which these distributions hold have been given by R. von Mises [4]. For a normal distribution, L. H. C. Tippett [7] has calculated the numerical values of the mean range and the first four moments of the range for sample sizes varying from 2 to 1000. He has shown that, for sample sizes exceeding 200, the correlation between the largest and the smallest observation may be neglected. Later, E. S. Pearson [5] has calculated the probability function of the range for small samples ($n = 2$ to 20) taken from a normal population. These calculations are very laborious. Recently, W. E. Deming [1] has applied the range to quality control.

The concepts "extremes", "range" and "midrange" allow a simple generalization. Let ${}_mx$ and x_m be the mth observation in increasing and in decreasing magnitude, henceforth called mth value "from below" and "from above". As long as the index m is small compared to the sample size n, the mth values under consideration are extremes. The difference and the sum of the mth extreme observations are called the mth *range* and the mth *midrange*. We will investigate the asymptotic distributions of the mth extremes, of the mth range, and of the mth midrange. Assuming that the number of observations is very large, the correlation between the largest and the smallest observation may be neglected. Then the mth range and the mth midrange are the difference and the sum of two independent variates, the mth extremes.

It was found that the distribution of the mth range is skew and the distribution of the mth midrange is of the generalized logistic type, which is symmetrical. For m increasing the distributions of the mth extremes, the mth ranges, and the mth midranges converge toward normality.

2. Generating functions of the mth extremes. Let $\varphi(x)$ be an initial continuous symmetrical distribution with mean zero; let u_m be the most probable mth value from above; let α_m be defined by

$$\alpha_m = \frac{n}{m}\,\varphi(u_m). \tag{1}$$

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Under conditions given in an earlier paper [3] the distributions $f_m(x_m)$ and ${}_mf({}_mx)$ of the mth extremes are, for large n and small m

$$f_m(x_m) = \alpha_m \frac{m^m}{(m-1)!} e^{-my_m - me^{-y_m}}; \qquad {}_mf({}_mx) = f_m(-x_m) \tag{2}$$

where

$$y_m = \alpha_m(x_m - u_m) \tag{3}$$

is a reduced extreme mth value.

The moment generating function $G_m(t)$ of the mth extreme value from above obtained from the preceding equations is

$$G_m(t) = \frac{m^m}{(m-1)!} e^{u_m t} \int_{-\infty}^{+\infty} e^{-my_m + y_m(t/\alpha_m) - me^{-y_m}} dy_m .$$

Introducing

$$e^{-y_m} = z$$

the integral becomes

$$\int_0^\infty z^{m-(t/\alpha_m)-1} e^{-mz} dz = \Gamma\left(m - \frac{t}{\alpha_m}\right) m^{-m+(t/\alpha_m)}$$

whence

$$G_m(t) = e^{u_m t} m^{t/\alpha_m} \Gamma\left(m - \frac{t}{\alpha_m}\right) \Big/ \Gamma(m). \tag{4}$$

To obtain the moments of the mth extreme value from above, the fundamental property of the Gamma function is used; whence

$$G_m(t) = e^{(t/\alpha_m)(u_m\alpha_m + lgm)} \prod_{\nu=1}^{m-1} \frac{m - \nu - \dfrac{t}{\alpha_m}}{m-\nu} \Gamma\left(1 - \frac{t}{\alpha_m}\right).$$

Finally, by reversing the order of multiplication the moment generating function of the mth extreme value from above becomes

$$G_m(t) = e^{(t/\alpha_m)(u_m\alpha_m + lgm)} \prod_{\nu=1}^{m-1} \left(1 - \frac{t}{\nu\alpha_m}\right) \Gamma\left(1 - \frac{t}{\alpha_m}\right). \tag{5}$$

The mean $\bar{x}_m$ of the mth value from above obtained from (5) is

$$\bar{x}_m = u_m + \frac{1}{\alpha_m}\left(lgm - \sum_{\nu=1}^{m-1} \frac{1}{\nu} + c\right) \tag{6}$$

where $c = .57722$ is Euler's constant.

The seminvariant generating function $L_m(t)$ of the mth extreme value from above becomes from (5) and (6)

$$L_m(t) = \frac{t}{\alpha_m}\left(\sum_{\nu=1}^{m-1} \frac{1}{\nu} - c\right) + \sum_{\nu=1}^{m-1} lg\left(1 - \frac{t}{\nu\alpha_m}\right) + lg\Gamma\left(1 - \frac{t}{\alpha_m}\right)$$

and after expansion

$$L_m(t) = \sum_2^\infty \frac{t^\nu}{\nu \alpha_m^\nu} S_{\nu,m} \tag{7}$$

where the sums

$$S_{\nu,m} = \sum_{k=m}^{\infty} \frac{1}{k^\nu}; \qquad \nu \geqq 2 \tag{8}$$

are obtained from the sums

$$S_\nu = \sum_{k=1}^{\infty} \frac{1}{k^\nu} \tag{8'}$$

which are known from the theory of the Gamma function. The numerical values of the seminvariants of the mth extreme value from above $\lambda_{\nu,m}$, being the coefficients of $t^\nu/\nu!$ in the expansion (7) may be calculated from a table of the sums $S_{\nu,m}$ given in an earlier paper [3].

From the generating functions (4) and (7) of the mth extreme value from above the moment generating function ${}_mG(t)$ and the seminvariant generating function ${}_mL(t)$ of the mth extreme value from below are obtained by the symmetric relation (2) as

$${}_mG(t) = G_m(-t); \quad {}_mL(t) = L_m(-t) \tag{9}$$

and the mean ${}_m\bar{x}$ of the mth extreme value from below is

$${}_m\bar{x} = -\bar{x}_m . \tag{6'}$$

The seminvariants $\lambda_{m,\nu}$ and ${}_m\lambda_\nu$ of the mth extreme value from above and from below are linked by

$$\alpha_m^\nu \lambda_{m,\nu} = (\nu - 1)! S_{\nu,m} = (-1)^\nu \alpha_{mm}^\nu \lambda_\nu . \tag{10}$$

The standard errors σ_m and ${}_m\sigma$ of the mth extreme values are

$$\alpha_m \sigma_m = \sqrt{S_{2,m}} = \alpha_{mm} \sigma . \tag{11}$$

This procedure for obtaining the moments of the mth extremes from their distribution (2) parallels closely that used by R. A. Fisher and L. C. H. Tippett [2] for obtaining the moments of the largest and smallest value. The special case $m = 1$ of the formulae (4), (9), (6), (6'), (11), and (10) leads to the moment generating functions, the means, the standard errors, and the higher seminvariants of the largest and of the smallest value given by these authors.

The two parameters u_m and α_m which exist in the distribution (2) of the mth extremes may be calculated from the observations by virtue of equations (6) and (11). Thus, the theoretical distributions (2) may be compared to observations even if the initial distribution $\varphi(x)$ is unknown. For increasing m, the distributions of the mth extremes were shown [3] to converge toward normal distributions, their means and standard deviations being given by (6), (6') and 11.

3. Generating functions of the mth range and the mth midrange. To obtain the characteristics for the mth range and the mth mid-range we state first some general properties of the sum and of the difference of two independent variates x and y, with means $\bar{x}$ and $\bar{y}$ and standard deviations σ_x and σ_y. Let the distributions be $\varphi(x)$ and $\psi(y)$, and let the generating functions of the seminvariants $\lambda_{x,\nu}$ and $\lambda_{y,\nu}$ be $L_x(t)$ and $L_y(t)$. We write

$$v = x + y; \qquad w = x - y \tag{12}$$

from the sum and the difference of the two variates. Then the means $\bar{v}$ and $\bar{w}$ and the variances σ_v^2 and σ_w^2 are

$$\bar{v} = \bar{x} + \bar{y}; \quad \bar{w} = \bar{x} - \bar{y}; \quad \sigma_v^2 = \sigma_x^2 + \sigma_y^2 = \sigma_w^2 \tag{13}$$

and the seminvariant generating functions $L_v(t)$ and $L_w(t)$ are

$$L_v(t) = L_x(t) + L_y(t); \quad L_w(t) = L_x(t) + L_y(-t). \tag{14}$$

The negative sign in the second equation (14) is obtained through the same well-known derivation as used for the first equation (14). Therefore, the even seminvariants of the sum are equal to the even seminvariants of the difference, whereas the odd seminvariants of the sum and of the difference are

$$\lambda_{v,2\nu+1} = \lambda_{x,2\nu+1} + \lambda_{y,2\nu+1}; \qquad \lambda_{w,2\nu+1} = \lambda_{x,2\nu+1} + (-1)^{2\nu+1}\lambda_{y,2\nu+1}.$$

If the distributions $\varphi(x)$ and $\psi(y)$ are symmetrical one to another in the sense

$$\psi(y) = \varphi(-x) \tag{15}$$

then the even seminvariants of the two variates x and y coincide in size and sign and the odd seminvariants coincide in size and differ in sign. Under the condition (15), the even seminvariants of the sum v and the even seminvariants of the difference w are twice the even seminvariants of the variates x or y. The odd seminvariants of the sum v are twice the odd seminvariants of the variate x, and the odd seminvariants of the difference w vanish.

We apply these properties to the mth extreme values and write x_m for x and ${}_mx$ for y. According to (2) the distribution of the mth extreme from above is symmetrical to the distribution of the mth extreme from below in the sense (15).

The mth range w_m and the mth mid-range v_m are defined by

$$w_m = x_m - {}_mx; \qquad v_m = x_m + {}_mx. \tag{12'}$$

The mean $\bar{w}_m$ of the mth range, the mean $\bar{v}_m$ of the mth mid-range and the respective variances $\sigma_{v_m}^2$ and $\sigma_{w_m}^2$ are, from (6), (6') and (11)

$$\bar{w}_m = 2\bar{x}_m; \qquad \bar{v}_m = 0; \qquad \sigma_{v_m}^2 = 2\sigma_m^2 = 2_m\sigma^2 = \frac{2S_{2,m}}{\alpha_m^2} = \sigma_{w_m}^2. \tag{13'}$$

The seminvariant generating functions $L_w(t, m)$ of the mth range and $L_v(t, m)$ of the mth mid-range obtained from (7) and (9) are

$$L_w(t, m) = 2\sum_{\nu=2}^{\infty} \frac{t^\nu}{\nu\alpha_m^\nu} S_{\nu,m} = 2L_m(t); \quad L_v(t, m) = 2\sum_{\nu=1}^{\infty} \frac{t^{2\nu} S_{2\nu,m}}{2\nu\alpha_m^{2\nu}}. \tag{14'}$$

The seminvariants of the mth range are twice the seminvariants of the mth extreme values from above. The even seminvariants of the mth mid-range and the even seminvariants of the mth range are twice the even seminvariants of the mth extremes. The odd seminvariants of the mth mid-range vanish. Therefore, the distribution of the mth range is skew, and the distribution of the mth midrange is symmetrical. From the convergence of the mth extremes towards normality it follows that the mth range and the mth mid-range tend also, for increasing indices m, toward normality.

The seminvariants of the range and the mid-range are obtained from (13′) and (14′) by putting $m = 1$ and omitting the index 1. Therefore, the standard errors σ_w and σ_v of the range and of the mid-range are

$$\alpha\sigma_w = \frac{\pi}{\sqrt{3}} = \sqrt{2}\,\sigma_n = \alpha\sigma_v \tag{13″}$$

where α is given by (1) and σ_n stands for the standard error of the first or the last value. The skewness $\beta_{1,w}$ of the range and the excess $\beta_2 - 3$ for the range and the mid-range

$$\beta_{1w} = .64928; \qquad \beta_{2,w} - 3 = 1.2 = \beta_{2v} - 3 \tag{16}$$

are only one half of the corresponding characteristics of the largest value. The distribution of the range is less skew and less concentrated toward the middle than the distribution of the largest value. The moments (13) and (16) of the range and of the midrange may also be obtained directly from Fisher's and Tippett's results [2] when independence of the two extremes is assumed. The numerical values (16) for the limiting distribution of the range differ considerably from the values

$$\beta_{1w} = .309; \qquad \beta_{2w} - 3 = .54 \tag{16′}$$

given by Tippett [7] for $n = 1000$ observations. For increasing n the approach of the distribution of the range toward its limiting distribution is very slow. This is reasonable as the approach of the distribution of the largest value toward the limiting distribution is also very slow.

Until now we considered symmetrical initial distributions. The case of an asymmetrical initial distribution may be dealt with briefly. The most probable mth extreme value from below ${}_mu$ differs then from $-u_m$ and

$${}_m\alpha = \frac{n}{m}\,\varphi({}_mu) \tag{1′}$$

differs from α_m. The distribution of the mth extreme value from below is for large n and small m [3]

$${}_mf({}_mx) = {}_m\alpha\,\frac{m^m}{(m-1)!}\,e^{{}_m\alpha\,{}_mx - me^{{}_m\alpha\,{}_mx}} \tag{2′}$$

where

$$(3') \qquad {}_m y = {}_m\alpha({}_m x - {}_m u)$$

is a reduced mth extreme value from below. The moment generating function ${}_mG(t)$ of the mth extreme value from below becomes

$$(4') \qquad {}_mG(t) = e^{{}_m u t}\, m^{-(t/{}_m\alpha)}\, \Gamma\left(m + \frac{t}{{}_m\alpha}\right) \Big/ \Gamma(m)$$

and the moment generating functions $G_w(t, m)$ and $G_v(t, m)$ of the mth range and of the mth mid-range of an asymmetrical distribution obtained from (4), (4') and in analogy to (14) are

$$(17) \qquad \begin{aligned} G_w(t, m) &= e^{t({}_m u - {}_1 u)}\, m^{t((1/\alpha_m)+(1/{}_m\alpha))}\, \Gamma\left(m - \frac{t}{\alpha_m}\right) \Gamma\left(m - \frac{t}{{}_m\alpha}\right) \Big/ \Gamma^2(m) \\ G_v(t, m) &= e^{t({}_m u + {}_1 u)}\, m^{t((1/\alpha_m)-(1/{}_m\alpha))}\, \Gamma\left(m - \frac{t}{\alpha_m}\right) \Gamma\left(m + \frac{t}{{}_m\alpha}\right) \Big/ \Gamma^2(m). \end{aligned}$$

Thus, the moments of the mth range and the mth mid-range may easily be computed even for an asymmetrical distribution.

4. The distribution of the mth midrange. In the following we return to a symmetrical distribution and establish directly the distribution $f(v_m)$ of the mth midrange. Then, the generating function (14') and the convergence toward normality will be verified.

From (12') the distribution of the mth midrange is

$$f(v_m) = \int_{-\infty}^{+\infty} f_m(x_m)\, {}_m f(v_m - x_m)\, dx_m .$$

Introducing (2), the equation is written

$$f(v_m) = \alpha_m \frac{m^{2m}}{(m-1)!^2} \int_{-\infty}^{+\infty} e^{-my_m + m\alpha_m(v_m - x_m + u_m) - me^{-y_m} - me^{\alpha_m(v_m - x_m + u_m)}}\, dy_m .$$

Using as before

$$e^{-y_m} = z$$

the integral becomes

$$e^{m\alpha_m v_m} \int_0^{\infty} z^{2m-1} e^{-mz(1+e^{\alpha_m v_m})}\, dz = \frac{e^{m\alpha_m v_m}(2m-1)!}{m^{2m}(1 + e^{\alpha_m v_m})^{2m}} .$$

The distribution of the mth midrange is therefore

$$(18) \qquad f(v_m) = \alpha_m \frac{(2m-1)!}{(m-1)!^2} \frac{e^{m\alpha_m v_m}}{(1 + e^{\alpha_m v_m})^{2m}} .$$

The distribution (18) will be shown to lead to the seminvariant generating function (14′). The generating function of the mth midrange obtained from (18) is

$$\frac{(2m-1)!}{(m-1)!^2}\int_{-\infty}^{\infty} e^{\alpha_m v_m(m+(t/\alpha_m))}(1+e^{\alpha_m v_m})^{-2m}\,d\alpha_m v_m .$$

Introducing

$$1+e^{\alpha_m v_m}=u^{-1}; \qquad du=-u^2 e^{\alpha_m v_m}\,d\alpha_m v_m$$

the integral is rewritten

$$\frac{(2m-1)!}{(m-1)!^2}\int_0^1 (1-u)^{m+(t/\alpha_m)-1}u^{-m-(t/\alpha_m)-1+2m-2}\,du$$

$$=\frac{\Gamma(2m)}{\Gamma^2(m)}\,\frac{\Gamma\left(m+\frac{t}{\alpha_m}\right)\Gamma\left(m-\frac{t}{\alpha_m}\right)}{\Gamma(2m)}.$$

Consequently, the moment generating function of the mth midrange is

$$G_v(t,m)=\frac{\Gamma\left(m+\frac{t}{\alpha_m}\right)\Gamma\left(m-\frac{t}{\alpha_m}\right)}{\Gamma^2(m)}. \tag{19}$$

This expression is identical with the product of the moment generating functions (4) and (9) of the extreme mth values. Therefore, equation (18) is the distribution, and equation (14′) is the seminvariant generating function of the mth midrange.

For $m=1$ the distribution (18) of the mid-range becomes the so-called logistic distribution which is commonly written

$$f(v)=\frac{\alpha e^{-\alpha v}}{(1+e^{-\alpha v})^2}. \tag{18′}$$

Accordingly, (18) may be called the *generalized logistic distribution*. The probability $F(v)$ of a value equal to, or less than, v obtained from (18′) is

$$F(v)=1/(1+e^{-\alpha v}). \tag{20}$$

Therefore the distribution $f(v)$ may be expressed by the probability $F(v)$ through

$$f(v)=\alpha F(v)(1-F(v)). \tag{18″}$$

Formula (20) considered as a growth function plays a rôle in population statistics where it was introduced about 100 years ago by Verhulst [8]. Recently, it has been widely used by R. Pearl [6]. In his treatment, the value v stands for the time, and the function $F(v)$ stands for the relative size of the population at time v compared to its alleged asymptotic size.

In the following, the influence of m on the distribution (18) is studied. The distribution $\phi_m(z)$ of the reduced mth mid-range

$$z = \alpha_m \sqrt{m}\, v \tag{21}$$

is, from (18)

$$\phi_m(z) = \frac{1}{\sqrt{m}} \frac{(2m-1)!}{(m-1)!^2} \frac{e^{z\sqrt{m}}}{(1+e^{z/\sqrt{m}})^{2m}}. \tag{22}$$

The probability density of the mean mth reduced mid-range increases with m. Indeed,

$$\frac{\phi_{m+1}(0)}{\phi_m(0)} = \frac{2m+1}{2\sqrt{m(m+1)}} > 1.$$

Therefore, the standard error of the reduced mth mid-range decreases with increasing index m. This is reasonable as the mth mid-range is an estimate of the median for the initial distribution. The larger m, the nearer are the mth extreme values to the median, and the better is the estimate.

To verify that the distribution (18) of the mth mid-range tends toward normality for large indices m equation (22) is rewritten

$$\lg \phi_m(z) = \lg \phi_m(0) + z\sqrt{m} - 2m \lg\left(\frac{1+e^{z/\sqrt{m}}}{2}\right).$$

Expansion of the exponential and the logarithm leads, if we neglect the third and higher powers of the deviation z, to

$$\lg \phi_m(z) = \lg \phi_m(0) + z\sqrt{m} - 2m\left(\frac{z}{2\sqrt{m}} + \frac{z^2}{4m} - \frac{z^2}{8m} + \cdots\right)$$

whence by virtue of (11)

$$f(v) = \text{Const}\, e^{-\alpha_m^2 m v_m^2/4}.$$

The distribution of the mth mid-range becomes normal for large indices m. The mean is zero, and the standard deviation is

$$\sigma_{v_m} = \frac{\sqrt{2}}{\alpha_m \sqrt{m}}. \tag{23}$$

This is in accordance with the statement (13′), as $S_{2,m}$ tends with increasing m toward $1/m$.

5. Summary. For initial symmetrical distributions the seminvariant generating functions (14′) of the mth range and the mth mid-range are obtained from the seminvariant generating functions (7) and (9) of the mth extreme values from above and from below. As the two mth extreme values are supposed to be independent our results hold only for very large, sample sizes. The

even seminvariants of the mth mid-range and of the mth range coincide. The distribution of the mth range is skew and the distribution of the mth mid-range is the generalized symmetrical logistic distribution. For increasing indices m the distributions of the mth extremes, the mth ranges, and the mth mid-ranges converge toward normality.

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NOTES

This section is devoted to brief research and expository articles on methodology and other short items.

NOTE ON ASYMPTOTIC VALUE OF PROBABILITY DISTRIBUTION OF SUM OF RANDOM VARIABLES WHICH ARE GREATER THAN A SET OF ARBITRARILY CHOSEN NUMBERS

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The purpose of this note is to present the following theorem which the author needed in connection with a computational problem. Since the theorem has general implications for statistical theory which do not seem to have been brought out heretofore, readers of this journal may find it of interest.

Theorem: *In Euclidean space of n dimensions with coordinates x_i $(i = 1, 2, \cdots n)$ let a_i be n constants whose sum is u_0, and consider a hyperplane*

$$x_1 + x_2 + \cdots + x_n = u, \qquad u \geq u_0. \tag{1}$$

Let k denote a positive constant, and $I(u)$ the $n-1$ fold integral defined by

$$I(u) = \int \cdots \int \exp.\left[-k\left(\sum_{i=1}^{n} x_i^2\right)\right] dx_1\, dx_2 \cdots dx_{n-1} \tag{2}$$

taken over that part of the hyperplane for which $x_i \geq a_i$. Then

$$\lim_{u\to\infty} \sqrt{n}\, e^{-ku^2/n} I(u) = (\pi/k)^{(n-1)/2}. \tag{3}$$

Proof: The integral may be reduced to another integral by reduction of the quadratic form

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n-1} x_i^2 + \left(u - \sum_{i=1}^{n-1} x_i\right)^2$$

to a sum of squares y_i^2 such that y_i does not involve x_j for $j < i$. Dropping the superscript $n-1$ in the notation for Σ and letting the subscript on Σ denote the least value of i involved in the sum, the expansion of this form may be written

$$2\sum_1 x_i^2 - 2u\sum_1 x_i + 2\sum_1 x_i x_j + u^2, \qquad i < j.$$

The transformation may be performed progressively as follows:

$$(\sqrt{2}\, x_1)^2 + 2\,(x_2 + x_3 + \cdots - u)\, x_1 + [(u - \sum_2 x_i)/\sqrt{2}]^2 = y_1^2$$

giving

$$y_1 = \sqrt{2}\, x_1 - (u - \sum_2 x_i)/\sqrt{2}.$$

Using the remainder of the terms involving x_2, one can complete the square with terms not involving x_2, and the value of y_2 is

$$y_2 = x_2\sqrt{3}/\sqrt{2} - (u - \sum_3 x_i)/\sqrt{6}.$$

Continuing until only terms involving x_{n-1} and u remain,

$$y_{n-1} = x_{n-1}\sqrt{n}/\sqrt{n-1} - u/\sqrt{n(n-1)}.$$

The remaining term in u will be found to be u^2/n.

Making the above transformation, the integral becomes

$$(4) \quad I(u) = [e^{-ku^2/n}/\sqrt{n}] \int \cdots \int \exp.\left[-k\left(\sum_{i=1}^{n-1} y_i^2\right)\right] dy_1\, dy_2 \cdots dy_{n-1}.$$

In order to fix the limits of integration on y_i it will be noted that the projection of the critical region of the hyperplane (1) upon the $n-1$ dimensional space in the original variables x_i delineates a region in that $n-1$ space bounded by the $n-1$ hyperplanes

$$x_i = a_i, \qquad i = 1, 2 \cdots, n-1$$

and the hyperplane

$$x_1 + x_2 + x_3 + \cdots + x_{n-1} = u - a_n.$$

Hence, if (2) is considered as an iterated integral with the integration performed in the order of the subscripts of x_i, the intervals of integration are

$$\begin{aligned}
a_1 &\leq x_1 \leq u - a_n - \sum_2 x_i \\
a_2 &\leq x_2 \leq u - a_n - a_1 - \sum_3 x_i \\
&\cdots\cdots \\
a_r &\leq x_r \leq u - \sum_{r+1} x_i - (u_0 - \sum_{r+1} a_i) \\
&\cdots\cdots \\
a_{n-1} &\leq x_{n-1} \leq u - u_0 + a_{n-1},
\end{aligned}$$

where it is recalled that $\sum_{r+1}$ denotes $\sum_{i=r+1}^{n-1}$.

Now transforming to y_i and using the general transformation equation

$$(5) \qquad y_r = x_r\sqrt{r+1}/\sqrt{r} - (u - \sum_{r+1} x_i)/\sqrt{(r+1)r}, \qquad 1 \leq r \leq n-1$$

where $\sum_n x_i = 0$ under definition of summation symbol noted above,

Lower limit of $y_r = -(u - \sum_{r+1} x_i)/\sqrt{(r+1)r} + a_r\sqrt{r+1}/\sqrt{r}$,

Upper limit of $y_r = (u - \sum_{r+1} x_i)\sqrt{r}/\sqrt{r+1} - (u_0 - \sum_{r+1} a_i)\sqrt{r+1}/\sqrt{r}$,

since

$$\sqrt{r+1}/\sqrt{r} - 1/\sqrt{(r+1)r} = \sqrt{r}/\sqrt{r+1}.$$

It is not difficult to show that

$$\sum_{r+1} x_i = C_{r+1}(y_i) + (n - 1 - r)u/n$$

where $C_{r+1}(y_i)$ denotes a linear combination of y_i for $i \geq r + 1$, which does not involve the variable u. In other words

$$u - \sum_{r+1} x_i = (r + 1)u/n - C_{r+1}(y_i).$$

Making this substitution, the limits on y_r are found to be

$$\text{Lower limit of } y_r = -(u/n)(\sqrt{r+1}/\sqrt{r}) + C_{r+1}(y_i)/\sqrt{(r+1)r} + a_r\sqrt{r+1}/\sqrt{r},$$

$$\text{Upper limit of } y_r = (u/n)(\sqrt{r(r+1)} - C_{r+1}(y_i)\sqrt{r}/\sqrt{r+1} - (u_0 - \sum_{r+1} a_i)\sqrt{r+1}/\sqrt{r}.$$

It will now be clear that as u becomes infinite, the limit of the integral in (4), will be the $n - 1$ fold integral taken over the whole of $n - 1$ space. This latter integral is easily evaluated to give $(\pi/k)^{(n-1)/2}$, and the theorem follows.

The following two corollaries, which are restatements of the theorem in less general form, bring out the implications for statistical theory.

COROLLARY 1. *With $k = 1/2$ define $F_n(u)$ by*

$$F_n(u) = (2\pi)^{-n/2} I(u). \tag{6}$$

The differential $F_n(u)du$ represents then the probability that n random variables x_i taken from a normally distributed population with zero mean and unit standard deviation, fall into a region

$$a_i \leq x_i \tag{7}$$

and have a sum with value in the neighborhood $(\pm\frac{1}{2}\, du)$ of u.

Recalling that u_0 is the value of u when each x_i has value a_i

$$P[a_i \leq x_i] = \int_{u_0}^{\infty} F_n(u)\, du \tag{8}$$

is the probability that all values of x fall into the region (7). *Denoting the normal probability function by $\phi(t)$, corollary 1 implies that*

$$\lim_{u \to \infty} \sqrt{n}\, F_n(u)/\phi(u/\sqrt{n}) = 1. \tag{9}$$

Since $\phi(u/\sqrt{n})du/\sqrt{n}$ represents the probability distribution of the sum of the n random variables when the condition (7) is removed, certain implications for the theory of statistics emerge. One of these is noted in the example given below. Corollary 1 can be stated in a different form as follows:

COROLLARY 2. *If $p_A(u)du$ denotes the probability that the sum of n random varia-*

bles from a normally distributed population be in the neighborhood ($\pm\frac{1}{2}\,du$) *of* u, *and* $p_B(u)$ *denotes the probability that the sum of* n *random variables from the same population that do fall in region* (7) *have a value in this neighborhood of* u, *it follows from* (9) *and the nature of the functions integrated that for arbitrarily small positive* δ, *a value of* u, *say* u' *can be found, sufficiently large, such that for all* $u \geq u'$,

$$(10) \qquad (1 - \delta)\, p_A(u) \leq p_B(u)\, P[a_i \leq x_i] \leq p_A(u).$$

Rate of convergence. The author having had occasion to compute $F_n(u)$ for values of n from 2 to 5, and $a_i = 0$, a table showing the rate of convergence of $F_n(u)$ to its limit for this range of n (and $a_i = 0$) is shown below. In this table the values of the ratios of the minimum values of u ($= u'$) to the standard deviation of u ($= \sqrt{n}$) are shown for a sequence of values of δ which approaches zero.

Rate of Convergence of $\sqrt{n}\,F_n(u)/\phi(u\sqrt{n})$ *to Unity for* $a_i = 0$

	Critical Ratio $u'/\sqrt{n}$			
δ	$n = 2$	$n = 3$	$n = 4$	$n = 5$
.5	0.67	1.34	1.92	2.46
.25	1.15	1.95	2.65	3.27
.1	1.64	2.60	3.40	4.11
.05	1.96	3.02	3.89	4.65
.01	2.58	3.85	4.87	5.76
.001	3.29	4.84	6.03	7.08

Example. An example showing the possible bearing of this theorem upon practical considerations of sampling is the following: If in a quality control problem, samples of size 4 were used, and a follow-up of samples which showed a large deviation of sample mean were pursued, a case of a particularly large deviation such as 4 would possibly receive special attention. With sample mean equal to $u/4$, the ratio of this mean to its standard deviation is $u/2$. Turning to the table, in column $n = 4$ it will be noted that the value of δ for $u'/2 = 4$ is somewhat less than .05. Hence from (9) and (10), for $u \geq u'$ ($u' = 8$, $n = 4$),

$$.95\,\phi(u/\sqrt{n})/\sqrt{n} < F_n(u) \leq \phi(u/\sqrt{n})/\sqrt{n}.$$

It follows that

$$.95\int_{u'}^{\infty} \phi(u/\sqrt{n})\,du/\sqrt{n} < \int_{u'}^{\infty} F_n(u)\,du \leq \int_{u'}^{\infty} \phi(u/\sqrt{n})\,du/\sqrt{n}.$$

If one now considers a set of random samples of size 4, the last integral on the right represents the expected proportion of the set which falls into the sub-set A for which $u \geq u'$ (and hence with deviation of mean relative to standard deviation of mean greater than $u'/\sqrt{n}$). The middle integral represents the expected proportion of the original set which falls into a sub-set B for which $u \geq u'$ *and* $x_i \geq 0$. It follows from the inequality on the left that the expected proportion of the sub-set A which falls into the sub-set B is greater than 95 per cent. Hence

one infers that the probability is greater than .95 that for a sample showing such a large deviation from the mean ($u/\sqrt{n} = 4, n = 4$) *all* the constituent elements will have deviations on the same side of the population mean. Thus if all the elements of the sample investigated are found to have deviations on the same side of the population mean, this could *not* be construed as *additional evidence* that the sample indicated an abnormal condition.

This conclusion is weaker than the facts of the example warrant, since it is based upon the *integral* of $F_n(u)$ from u' to infinity. Unfortunately the author does not have data available on the rate of convergence of these integrals.

NOTE ON A MATRIC THEOREM OF A. T. CRAIG

By Harold Hotelling

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An extremely elegant theorem given recently by A. T. Craig[1] and applied by him to establish a further theorem on independent χ^2 distributions may be stated as follows:

If A and B are the symmetric matrices of two homogeneous quadratic forms in n variates which are normally and independently distributed with zero means and unit variances, a necessary and sufficient condition for the independence in probability of these two forms is that $AB = 0$.

The proof given that the condition is sufficient is adequate, but Craig's treatment of its necessity consists essentially in its assertion. In view of the growing interest in such quadratic forms, for example in connection with serial correlation, the neatness of this theorem is likely to lead to a wide usefulness. It therefore seems worth while to give a complete proof of the necessity condition.

The form with matrix A is denoted by Q_1 and that with matrix B by Q_2. The characteristic functions, if defined as $Ee^{\frac{1}{2}\lambda Q_1}$ and $Ee^{\frac{1}{2}\mu Q_2}$, are respectively the reciprocals of the square roots of the determinants of the matrices $1 - \lambda A$ and $1 - \mu B$, while the characteristic function for Q_1 and Q_2 together, $Ee^{\frac{1}{2}(\lambda Q_1 + \mu Q_2)}$, is the reciprocal of the square root of the determinant of $1 - \lambda A - \mu B$. A necessary and sufficient condition for independence is therefore that

$$| 1 - \lambda A | \cdot | 1 - \mu B | \equiv | 1 - \lambda A - \mu B |$$

shall hold identically for all values of λ and μ. Since the determinant of the product of two matrices is the product of their determinants, the left member is the same as

$$| 1 - \lambda A - \mu B + \lambda\mu AB | .$$

From this it is immediately obvious that $AB = 0$ implies the independence of the two forms. The converse will now be proved.

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[1] "Note on the independence of certain quadratic forms," *Annals of Math. Stat.*, Vol. 14 (1943), pp. 195–197.

We shall assume therefore that Q_1 and Q_2 are independent, so that the identity holds, and prove that $AB = 0$.

It must not be supposed that Q_1 and Q_2 can by the same linear transformation be reduced to forms in which product terms are absent and only terms in the squares of the variates appear. The available theorems[2] leading to this canonical form require that at least one of the quadratic forms be non-singular. But it is of the essence of the present situation that both Q_1 and Q_2 be singular, since this is implied by $AB = 0$. It does not appear possible, for example, to reduce to this canonical form the pair $x_1^2 + x_2^2$ and $x_1^2 + 2x_1x_3$.

Nevertheless a real orthogonal transformation can be found reducing Q_1 to

$$d_1x_1^2 + \cdots + d_rx_r^2,$$

where r is the rank of Q_1. Thus there exists an orthogonal P such that $A = PLP^{-1}$ and $B = PMP^{-1}$, where L and M, when partitioned so as to separate the rows and columns into successive groups of r and $n - r$, are of the forms

$$L = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right], \qquad M = \left[\begin{array}{c|c} E & C \\ \hline C' & F \end{array}\right].$$

Here D stands for an r-rowed diagonal matrix having $d_1 \cdots d_r$ in its diagonal, 0 for various matrices whose elements are all zero, and E, C and F for arbitrary matrices of appropriate dimensions. Then

$$| 1 - \lambda A | = | P(1 - \lambda L)P^{-1} | = | P | \cdot | 1 - \lambda L | \cdot | P^{-1} | = | 1 - \lambda L |,$$

and in the same way,

$$| 1 - \mu B | = | 1 - \mu M |$$

and

$$| 1 - \lambda A - \mu B | = | 1 - \lambda L - \mu M |.$$

We thus have

$$| 1 - \lambda L | \cdot | 1 - \mu M | \equiv | 1 - \lambda L - \mu M |.$$

From this identity it follows that a pair of forms Q_1^* and Q_2^*, quadratic in a set of variates normally and independently distributed with zero means and unit variances, and having matrices L and M respectively, are independent.

Since $AB = PLMP^{-1}$, the theorem will be proved if we can show that $LM = 0$. Let

$$M_1 = \left[\begin{array}{c|c} E & C \\ \hline C' & 0 \end{array}\right], \qquad M_2 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & F \end{array}\right],$$

so that $M = M_1 + M_2$. Since obviously $LM_2 = 0$, we need only to show that $LM_1 = 0$.

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[2] E.g., Bôcher, *Introduction to Higher Algebra*, pp. 169, 305.

Let $Q_2^* = Q' + Q''$, where Q' and Q'' are quadratic forms in the particular normally distributed variates considered, and have matrices M_1 and M_2 respectively. Since Q'' obviously does not involve any of the variates occurring in Q_1^* and since all the variates are independent it follows that Q_1^* is independent of Q''. Since it has been shown above that Q_1^* is also independent of Q_2^*, it must be independent of the difference $Q_2^* - Q'' = Q'$. Therefore

$$| 1 - \lambda L | \cdot | 1 - \mu M_1 | \equiv | 1 - \lambda L - \mu M_1 | .$$

We have:

$$| 1 - \lambda L | = \prod_{i=1}^{r} (1 - \lambda d_i).$$

Also,

$$1 - \lambda L - \mu M_1 = \left[\begin{array}{c|c} 1 - \lambda D - \mu E & -\mu C \\ \hline -\mu C' & 1 \end{array}\right].$$

Consequently equating the terms of highest degree in λ on the two sides of the identity

$$\Pi(1 - \lambda d_i) \, | 1 - \mu M_1 | \equiv | 1 - \lambda L - \mu M_1 |$$

yields the identity in μ,

$$| 1 - \mu M_1 | \equiv 1,$$

or upon putting $\mu = 1/x$,

$$| M_1 - x | = (-x)^n.$$

Hence all the latent roots of the real symmetric matrix M_1 are zero. Now for a symmetric matrix the sum of the squares of the latent roots equals the sum of the squares of the elements, since both equal the trace of the square of the matrix. Therefore $M_1 = 0$. Consequently $LM_1 = 0$ and the theorem is established.

Since $M_1 = 0$, the following further result is obvious:

Two independent quadratic forms in a set of variates normally and independently distributed with zero means and a common variance can by a transformation be reduced to two forms having no variate in common.

But one of the disjunct sets of variates in the forms as thus reduced is not necessarily independent of the other set. For example, if x_1, x_2, x_3, x_4 are normally distributed with equal variances and any fixed non-vanishing correlation, the same in each of the six pairs, the sets (x_1, x_2) and (x_3, x_4) are not independent of each other, but the forms $(x_1 + x_2)^2$ and $(x_3 - x_4)^2$ are, since $x_1 + x_2$ is uncorrelated with $x_3 - x_4$.

I am indebted to Professor E. R. Lorch in connection with the preparation of the present note.

A NOTE ON THE BEHRENS-FISHER PROBLEM

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A commonly occurring problem of statistical inference is the comparison of the means of two normal universes when the ratio of their variances is unknown. Let $(x_1, \cdots, x_m)$ be a random sample from one normal population with mean α and variance μ, and $(y_1, \cdots, y_n)$, a random sample from another with mean β and variance ν. The problem is then that of making statistical inferences about the difference δ of the means, $\delta = \alpha - \beta$, when the ratio μ/ν is unknown. Convenient tests and confidence intervals are available if one can find a linear form L and a quadratic form Q in the vector $(x_1, \cdots, x_m, y_1, \cdots, y_n)$ with coefficients independent of the unknown parameters α, β, μ, ν, such that for some positive integer k, the quotient

$$(1) \qquad (L - \delta)/(Q/k)^{\frac{1}{2}}$$

has the t-distribution with k degrees of freedom. For this it is sufficient that the following conditions be satisfied for *all* values of the parameters: (*i*) *L and Q are independently distributed,* (*ii*)[1] *$E(L) = \delta$,* (*iii*) *Q/σ^2 has the χ^2-distribution with k degrees of freedom, where σ^2 is the variance of L.*

In a recent paper [1] the author investigated the Behrens-Fisher problem as delimited by the above three conditions,[2] and among other results arrived at a simple solution. This solution however does not have the property that the quotient (1) is symmetric, whereby in this note we shall mean the following: A function of the samples and parameters will be called symmetric if it is invariant under permutations of the x's among themselves and of the y's among themselves. Let us therefore formulate condition (*iv*): *the quotient* (1) *is symmetric.* Since (*iv*) would be extremely desirable, both for practical and theoretical reasons, and since the author has received several inquiries on this matter, it is considered worth while to outline a proof that conditions (*ii*) and (*iii*) imply that (*iv*) cannot be satisfied, in other words, there exists no "symmetric solution" of the Behrens-Fisher problem within the framework we have imposed. Perhaps this is a simple example of a larger class of problems in which an approach, natural in the light of past developments, forces us to an asymmetric solution.

Suppose (*iv*) is satisfied. By substituting special values for the vector $(x_1, \cdots, x_m, y_1, \cdots, y_n)$ and then making permutations allowed by (*iv*) we find that L and Q must be of the form

$$(2) \qquad L = c_1 \sum_i x_i + c_2 \sum_j y_j ,$$

$$(3) \qquad Q = c_3 \sum_i x_i^2 + c_4 \sum_{i \neq i'} x_i x_{i'} + c_5 \sum_j y_j^2 + c_6 \sum_{j \neq j'} y_j y_{j'} + c_7 \sum_{i,j} x_i y_j ,$$

---

[1] $E(f)$ denotes the expected value of f.

[2] Although these conditions appear simpler than those in [1] they may be shown equivalent.

where all c's are independent of parameters, and the range of indices i, i' is from 1 to m, the range of j, j' from 1 to n.

Condition (*ii*) requires that

$$E(L) = \alpha - \beta. \tag{4}$$

But from (2),

$$E(L) = c_1 m\alpha + c_2 n\beta. \tag{5}$$

Since (4) and (5) must be satisfied identically in α, β, it follows that $c_1 = 1/m$, $c_2 = -1/n$, and thus the variance of L is

$$\sigma^2 = \mu/m + \nu/n. \tag{6}$$

Because of condition (*iii*) we must have $E(Q/\sigma^2) = k$, and combining this with (6), we have

$$E(Q) = k(\mu/m + \nu/n). \tag{7}$$

However, from (3),

$$E(Q) = c_3 m(\mu + \alpha^2) + c_4(m^2 - m)\alpha^2 + c_5 n(\nu + \beta^2) + c_6(n^2 - n)\beta^2 + c_7 mn\alpha\beta. \tag{8}$$

Equating (7) and (8) gives us an identity in α, β, μ, ν from which we can determine the c's, and after putting these back in (3) we find that the result may be written

$$Q = k[S_x/(m^2 - m) + S_y/(n^2 - n)], \tag{9}$$

where

$$S_x = \sum_i (x_i - \bar{x})^2, \qquad \bar{x} = \sum_i x_i/m,$$

$$S_y = \sum_j (y_j - \bar{y})^2, \qquad \bar{y} = \sum_j y_j/n.$$

The last step of the proof consists of showing that Q defined by (9) violates (*iii*). Write $u_1 = S_x/\mu$, $u_2 = S_y/\nu$. Then u_1 and u_2 have independent χ^2-distributions. Now (9) states that $u = Q/\sigma^2$ is of the form $u = a_1u_1 + a_2u_2$, where a_1 and a_2 are constants. Let $\phi(t), \phi_1(t), \phi_2(t)$ be the respective characteristic functions of u, u_1, u_2. Then $\phi(t) = \phi_1(a_1t)\phi_2(a_2t)$ because u_1 and u_2 are statistically independent. Since the characteristic function of a χ^2-variable with r degrees of freedom is $(1 - 2it)^{-\frac{1}{2}r}$, it is evident that u has a χ^2-distribution if and only if $a_1 = a_2 = 1$. From (9),

$$a_1/a_2 = [\mu/(m^2 - m)]/[\nu/(n^2 - n)],$$

and thus a necessary condition (it is also sufficient) for Q/σ^2 to have a χ^2-distribution is

$$\mu/\nu = (m^2 - m)/(n^2 - n). \tag{10}$$

But (*iii*) states that Q/σ^2 has a χ^2-distribution for *all* parameter values. This contradiction completes the proof.

We remark in closing that we have at hand a counter example of practical interest to the statement found in several statistics texts that if z is a normal variable with zero mean and v is an independent unbiased quadratic estimate of the variance of z, then $z/v^{\frac{1}{2}}$ has a t-distribution. The counter example consists of taking $z = \bar{x} - \bar{y} - \delta$ and $v = Q/k$ defined by (9). It may be shown that $z/v^{\frac{1}{2}}$ does not have a t-distribution except in the trivial case (10).

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ON MULTIPLE MATCHING WITH ONE VARIABLE DECK

T. N. E. GREVILLE

Bureau of the Census

The problem of card matching has been considered by a number of writers. A complete bibliography has been given by Battin [1], who also published the most general treatment of the subject to date, dealing with the simultaneous matching of any number of decks of arbitrary composition. He considers, however, only the case in which the order of every deck is variable, all possible permutations being equally likely. Some interest attaches to the case in which all the decks but one have fixed orders in relation to one another, especially in connection with radio experiments in telepathy, where a large number of subjects simultaneously attempt to call the same target.

The simplest case is that in which the target for each trial is chosen at random, independently of the other trials. If the target is to be selected from s possibilities, and if p_i denotes the probability that the ith possibility will be chosen as the target, while m_i denotes the number of subjects who call the ith possibility, then the mean value of h, the number of correct calls is, of course,

$$M_h = \sum_{i=1}^{s} p_i m_i , \tag{1}$$

and the variance is

$$V_h = \sum_{i=1}^{s} p_i m_i^2 - M_h^2 . \tag{2}$$

Evidently, the mean number of hits for a succession of trials is the sum of the means for the individual trials, and the variance is the sum of the variances.

A slightly more difficult problem is presented when the target series is a true "deck": that is, when its composition is determined in advance, only the order being left to chance. Let n denote the number of trials and $n_i (i = 1, 2, \cdots, s)$

the number of targets of the ith kind so that $\sum_{i=1}^{s} n_i = n$. Also let $m_{ij} (j = 1, 2, \cdots, n)$ denote the number of subjects who call the ith possibility on the jth trial, and let m_i denote the total number of such calls in all the n trials, so that $m_i = \sum_{j=1}^{n} m_{ij}$. Then, using the ingenious type of counting function introduced by Wilks [6] and improved by Battin [1], we define

$$\Phi = \frac{1}{N} \prod_{j=1}^{n} \left(\sum_{i=1}^{s} x_i e^{m_{ij}\theta} \right),$$

where $N = n! \Big/ \prod_{i=1}^{s} n_i!$. We also define an operator K_{n_i} such that if $u = u(x_1, x_2, \cdots, x_s)$, then $K_{n_i} u$ is the coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_s^{n_s}$ in u. Then, if h denotes the number of hits

$$P(h = r) = \text{coefficient of } e^{r\theta} \text{ in } K_{n_i}\Phi.$$

Also,

$$E(h^p) = K_{n_i} \frac{\partial^p \Phi}{\partial \theta^p}\bigg|_{\theta=0}.$$

It follows immediately that

$$M_h = \frac{1}{n} \sum_{i=1}^{s} n_i m_i = \sum_{i=1}^{s} p_i m_i , \tag{3}$$

where p_i is written for n_i/n. Similarly, it can be shown that

$$E(h^2) = \frac{1}{n} \sum_{i=1}^{s} \left(n_i \sum_{j=1}^{n} m_{ij}^2 \right) + \frac{1}{n(n-1)} \sum_{i=1}^{s} \left[n_i(n_i - 1) \sum_{\substack{j,k=1 \\ j \neq k}}^{n} m_{ij} m_{ik} \right] + \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{s} \left[n_i n_j \sum_{\substack{k,l=1 \\ k \neq l}}^{n} m_{ik} m_{jl} \right].$$

If we write $\lambda_{ij} = \sum_{k=1}^{n} m_{ik} m_{jk}$, it is found that

$$V_h = \frac{1}{n-1} \left[\sum_{i=1}^{s} p_i(n\lambda_{ii} - m_i^2) - \sum_{i,j=1}^{s} p_i p_j (n\lambda_{ij} - m_i m_j) \right]. \tag{4}$$

It should be noted that it is not necessary that the total number of subjects be the same for each trial.

Certain special cases are of interest. When there is just one subject for each target, as in the ordinary matching of two decks, $\lambda_{ij} = m_i$ for $i = j$, and 0 otherwise; and $\Sigma_i m_i = n$. Then, if π_i denotes m_i/n and $\rho_i = 1 - \pi_i$, and if $q_i = 1 - p_i$,

$$M_h = n\Sigma_i p_i \pi_i \quad \text{and} \quad V_h = \frac{n^2}{n-1} [\Sigma_i p_i q_i \pi_i \rho_i + \sum_{i \neq j} p_i p_j \pi_i \pi_j], \tag{5}$$

a compact expression which is equivalent to the forms previously given by Stevens [4, 5], Greenwood [2], Battin [1], and the author [3].

On the other hand, if the different kinds of targets occur with equal frequency, so that every $p_i = 1/s$ and every $q_i = (s-1)/s$; and if $\nu_k = \Sigma_i m_{ik}$ and $\nu = \Sigma_i m_i = \Sigma_k \nu_k$, the expression (4) becomes

$$V_h = \frac{n}{s^2(n-1)} [\nu^2 - n\Sigma_k \nu_k^2 + s\Sigma_i(n\lambda_{ii} - m_i^2)]. \tag{6}$$

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[1] I. L. Battin, "On the problem of multiple matching," *Annals of Math. Stat.*, Vol. 13 (1942), pp. 294–305.

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[3] T. N. E. Greville, "The frequency distribution of a general matching problem," *Annals of Math. Stat.*, Vol. 12 (1941), pp. 350–354.

[4] W. L. Stevens, "Distribution of entries in a contingency table," *Annals of Eugenics*, Vol. 8 (1938), pp. 238–244.

[5] W. L. Stevens, "Tests of significance for extra sensory perception data," *Psychol. Review*, Vol. 46 (1939), pp. 142–150.

[6] S. S. Wilks, "Statistical aspects of experiments in telepathy," a lecture delivered to the Galois Institute of Mathematics, Long Island University, Dec. 4, 1937.

BOOK REVIEW

"Student's" Collected Papers. Edited by E. S. PEARSON and JOHN WISHART. Biometrika Office, London. pp. xiv + 224. 1942. 15 shillings.

Since the question is still asked from time to time, it may be well to state that 'Student' was the pen name of William Sealy Gosset. After taking his degree in mathematics and natural sciences at Oxford, Student worked as a brewer for the well-known firm of Messrs. Guinness—in Ireland from 1899 till 1935 and thereafter in London as head of the new Guinness brewery established there. He died in 1937 at the age of 61. Between 1907 and 1937 he published twenty-one papers and a few notes; these are issued in the present volume as a tribute from a group of his relatives and friends.

Student's name is almost universally attached to a single discovery, the t-test, which requires for a complete proof more powerful mathematical methods than he devised. These facts may cause his papers to be regarded as museum pieces by people who have not read them. Actually, in many respects no better model than Student could be suggested for a young statistician today. Since we sometimes speak derisively of courses of lectures in statistics where the methods and ideas are "20 years out of date", it is interesting to find that many of the fundamental ideas considered most in need of emphasis by forward-looking teachers today were in fact emphasized by Student in his writings over 30 years ago.

For example, his classical paper on the t-test, published in 1908, opens as follows: "Any experiment may be regarded as forming an individual of a 'population' of experiments which might be performed under the same conditions. A series of experiments is a sample drawn from this population.

Now any series of experiments is only of value in so far as it enables us to form a judgment as to the statistical constants of the population to which the experiments belong".

The idea is elaborated in a second paper published in the same year. "Note that the indefinitely large population need not actually exist. In Mr. Hooker's case his sample was 21 years of farming under modern conditions in England, and included all the years about which information was obtainable. Probably it could not actually have been made much larger without loss of homogeneity, due to the mixing with farming under conditions not modern; but one can imagine the population indefinitely increased and the 21 years to be a sample from this." We note here a further observation, not always appreciated, that in some lines of research samples which are both large and *homogeneous* do not exist. A clear conception of the relation between sample and population is evident in all his work: one further quotation, the summary from his 1926 paper, *Mathematics and Agronomy*, will suffice. "To sum up, in planning agronomic experiments use plenty of replications and make quite sure your results are capable of being considered to be a random sample of the population about which you wish to draw conclusions."

Another noticeable feature of Student's work is his concern about the extent to which the mathematical model underlying his statistical techniques really represents the facts of the data. Further, he is always ready to give his opinion, from considered thought, as to whether the discrepancies between the model and the facts will affect the conclusions materially. In his first paper, *On the error of counting with a haemacytometer*, he shows that the Poisson distribution applies to the total number of cells or corpuscles which are found in a division of the haemacytometer field. He is careful, however, to point out three ways in which deviations from the Poisson law may occur in practice: (i) there may be interference between the particles, though this is considered unlikely in high dilutions (ii) there may be clumping and (iii) (probably the most important, in his view) the drops taken out for counting may not represent the bulk of the liquid, giving rise to an additional sampling error superimposed on the Poisson error. In a later paper *An explanation of deviations from Poisson's law in practice* (*1919*), he makes a critical examination of the assumptions necessary for Poisson's distribution, and shows the type of distribution to be expected from the invalidity of one or another of the assumptions.

Similarly, in the opening paragraphs of the 1908 paper on the t-test, he notes that the assumption of normality in the parent distribution may not hold in practice. Then follows his opinion on the importance of this assumption: "It appears probable that the deviation from normality must be very extreme to lead to serious error."

His earliest papers deal with attempts to find several exact small-sample distributions for which he and other workers in Britain felt a need. Lacking the mastery of probability theory necessary for obtaining a rigorous solution, he used a combination of mathematics, intelligent guesswork and inferences from repeated samplings which he drew. For example, the steps in his development of the t-distribution were as follows:

(i) The distribution of the mean square, s^2, being unknown to British statisticians at that time, had first to be found. Student calculated the first four moments by algebraic methods. Noticing that the condition for fitting a Pearson Type III curve ($2\beta_2 - 3\beta_1 - 6 = 0$) held exactly, he inferred correctly that s^2 followed a type III curve.

(ii) He was aware that the joint distribution of s^2 and the sample mean $\bar{x}$ was necessary for a solution of the problem and further that the form of the joint distribution depended on the correlation between $\bar{x}$ and s^2. Having shown by algebra that s^2 is uncorrelated with either $\bar{x}$ or $\bar{x}^2$, he assumed (without further comment) that the distributions of $\bar{x}$ and s^2 were independent.

(iii) The final step to the t-distribution was performed rigorously.

(iv) The result was then checked against empirical data. He used the heights of 3000 criminals, which provided 750 samples of 4. Another set of 750 samples was supplied by the lengths of the left middle fingers of the same criminals.

In 1908 Student also published the distribution of the sample correlation coefficient r, in a bivariate normal population in which ρ is zero. His interest in

the problem arose from a meeting of the Royal Statistical Society, in which an inquiry was raised about the significance of correlation coefficients derived from small numbers of cases. The authorities present, Yule, Hooker and Edgeworth, ventured their opinion on a specific instance, but had no general answer. Accordingly, Student set out to find the general distribution of r, without assuming ρ zero. His mathematics did not take him far; providing the solution only for samples of 2, where the frequency is zero except at -1 and $+1$. In order to learn what empirical results could teach about the form of the distribution, he then drew four sets of 750 samples, the sets having $\rho = 0$, $n = 4$; $\rho = 0$, $n = 8$; $\rho = .66$, $n = 4$; and $\rho = .66$, $n = 8$, respectively.

He considered first the set having $\rho = 0, n = 4$. Now, for samples of 4, the exact distribution, $C(1 - r^2)^{\frac{1}{2}(n-4)}$, reduces to a rectangular distribution. For his purpose, it would seem that Student had been unlucky in his choice of $n = 4$, since the rectangular distribution appears to furnish little or no clue as to the general form of distribution when n differs from 4. Actually, things could not have turned out better. Referring, as in the case of the distribution of s^2, to Pearson's curves, he selected a type II on account of the limited range of the distribution of r. The fitted curve was found to be $C\left(1 - \frac{r^2}{1.076}\right)^{0.272}$. This suggested to Student that the true form ought to be $C(1-r^2)^0$, giving a rectangular distribution; the general form was guessed as $C(1 - r^2)^{\frac{1}{2}(n-4)}$. The samples of 8 were then used merely as corroborative material. One wonders how Student would have fared, both in this case and in the case of s^2, if the Pearson system of curves had not been familiar to him. The method did not of course provide the much more complex distribution when ρ is not zero; however, Student was able to write down several important properties of the distribution for the guidance of mathematical statisticians who might be interested in the problem.

By similar methods he attempted (1921) to find the frequency distribution and the efficiency of Spearman's rank correlation coefficient. Much later (1936), he remarked in connection with this coefficient that, though it was not fully efficient, "it was so simple that when playing with *other* people's figures, for instance on a railway journey, it was the obvious one to use."

His paper, *The elimination of spurious correlation due to position in time and space*, though brief and incomplete in its treatment of the problem, established him as one of the pioneers in the use of the variate-difference method. His suggestion was to correlate the nth differences of x with those of y for $n = 1, 2, \cdots$ until a point was reached where the correlation ceased to change. This limiting correlation would be free from spurious effects due to trend factors. Such methods, he remarks, help to show "whether there really *is* a close connexion between the female cancer death rate and the quantity of imported apples consumed per head."

His association with field experiments conducted in Ireland for the purpose of developing high yielding varieties of barley suitable for brewing led to an interest in the design of experiments, to which he devoted seven papers. His philosophy

on this subject, though it appeared to be increasingly out of harmony with the philosophy underlying Fisher's methods, was a natural outgrowth of Student's general attitude towards the applications of statistical techniques. He believed that, from general experience of soil fertility patterns plus local knowledge of the history and topography of the particular field in question, one could construct a systematic design which would be more accurate than the most appropriate randomized design. Secondly, although systematic designs rely on nature, as it were, to supply the random element necessary for the application of probability theory, he believed, again from knowledge of field conditions, that the discrepancies between the mathematical model and the true fertility pattern would not seriously vitiate the estimate of error. How far he would have carried the advocacy of systematic arrangements is not clear. Nearly all of his discussions refer to arrangements for comparing only two varieties. Moreover, he stressed repeatedly the necessity for carrying out such trials over several seasons and at a considerable number of places, in order to sample weather and soil variations. In such cases the statistical significance of a result at a single place did not interest Student greatly.

A few quotations of his opinions on more detailed questions illustrate his understanding of the attitude of the farmer and agronomist and his direct mode of expression. Speaking of experimentation with large plots, he says "it has the advantage that the farmer, who always has a healthy contempt for gardening, may pay some attention to the results." On adjustments to yield data to allow for variations in soil fertility, he writes: "There is a great disadvantage in correcting any figures for position, inasmuch as it savors of cooking, and besides the corrected figures do not represent anything real." Long experience had made him more and more convinced that a serial correlation is inevitable in routine chemical analyses; accordingly, he advises: "The chemist who wishes to impress his clients will therefore arrange to do repetition analyses as nearly as possible at the same time, but if he wishes to diminish his real error he will separate them by as wide an interval of time as possible."

Re-reading of Student's papers has been a keen pleasure. It is hoped that the volume will enjoy a wide circulation amongst statisticians.

W. G. COCHRAN

NEWS AND NOTICES

Readers are invited to submit to the Secretary of the Institute news items of interest

Personal Items

Dr. P. H. Anderson, formerly of War Production Board in Cleveland, is now associated with the Bureau of Foreign and Domestic Commerce in Washington, D. C.

Dr. T. A. Bancroft, formerly instructor at Iowa State College, has been promoted to an assistant professorship.

Jarvis Barnes is now Supervisor of Statistics for the Atlanta Board of Education.

Associate Professor W. D. Baten has been promoted to the rank of Professor of Mathematics and Research Professor at the Michigan Agricultural Experiment Station of Michigan State College.

Isaac L. Battin is now Assistant Professor of Mathematics at Drew University.

Robert E. Bechhofer has been promoted from Assistant Statistician to Associate Statistician at Aberdeen Proving Ground.

Blair M. Bennett is now Associate Mathematician with the National Bureau of Standards at Washington.

Assistant Professor Thomas A. Bickerstaff on leave from the University of Mississippi, is in attendance at the University of Michigan.

Dr. Ernest E. Blanche, formerly with Curtiss-Wright, is now Principal Statistician, Chief of Operations with Statistics Division, Foreign Economic Administration.

Assistant Professor Frances L. Campbell, on leave from her position as Dean of Women at George Pepperdine College, is studying at the University of Michigan.

Professor Haskell B. Curry, on leave from Pennsylvania State College, is now Mathematician at the Applied Physics Laboratory, Johns Hopkins University.

Dr. A. H. Copeland of the University of Michigan has been promoted to a professorship.

Dr. Gerald J. Cox, formerly with the National Research Council, is now Research Chemist with Corn Products Refining Company, Argo, Illinois.

Assistant Professor Paul Dressel of Michigan State College has been appointed Chairman of the Board of Examiners and Director of Counselling.

William F. Elkin has accepted a position as Social Science Analyst, Vital Statistics Division, Bureau of the Census at Washington.

Mark W. Eudey, on leave from the University of California, is serving as Operations Analyst with the Ninth Tactical Air Command.

Associate Professor Herbert P. Evans has been promoted to the rank of Professor of Mathematics at the University of Wisconsin.

Raymond H. Fadner, formerly with the OCD, is now Statistician for the United Nations Relief and Rehabilitation Administration.

Dr. Nicholas A. Fattu is now with the College Entrance Examination Board at Princeton.

Evelyn Fix is now Lecturer in Mathematics and Research Assistant at the Statistics Laboratory of the University of California.

Assistant Professor Harold A. Freeman has been promoted to the rank of Associate Professor of Statistics at Massachusetts Institute of Technology.

Milton Friedman, formerly with the Treasury Department, is now with the Columbia University Statistical Research Group.

E. L. Godfrey is now instructor in Mathematics and Physics at the Northwest Missouri State Teachers College at Maryville.

Harry H. Goode, formerly Statistician with the New York Department of Health, is now Research Associate in Mathematics at Tufts College.

Mrs. Dorothy K. Gottfried, formerly Assistant Statistician with the OPA, is now Research Analyst with the United States Army Signal Corps at Newark.

Trygve Haavelmo, formerly with the Norwegian Shipping and Trade Mission, is now Commercial Secretary at the Norwegian Embassy at Washington.

M. H. Henry, formerly Assistant Statistician with the Michigan Department of Social Welfare, is now Cost Analyst for the Fisher Body Corporation at Detroit.

Dr. Robert W. B. Jackson has been promoted to the rank of Assistant Professor and Assistant Director of the Department of Educational Research at the Ontario College of Education.

Dr. Rachel M. Jenss, formerly Economic Analyst with the Treasury Department, is now with the State Department in the capacity of Assistant Chief, Cost of Living Division, Foreign Service Administration.

Dr. Irving Kaplansky is Assistant Research Mathematician at Columbia University.

Leo Katz, formerly Statistician with the Michigan Department of Labor and Industry, is now with the Aircraft Development Section of General Motors Corporation at Detroit.

Lila F. Knudsen is now Statistician with the Food and Drug Administration.

Dr. Richard L. Kozelka has been promoted to the rank of Professor of Economics and Statistics and Acting Dean, School of Business Administration at the University of Minnesota.

Dr. George M. Kuznets has been promoted to the rank of Assistant Professor of Agricultural Economics at the University of California.

Max LeLeiko, formerly at the University of North Carolina, is now an instructor at Rutgers University.

Assistant Professor M. S. Macphail of Acadia University has been promoted to an associate professorship.

Dr. William G. Madow, formerly with the Bureau of Agricultural Economics, is now Statistician with the Census Bureau.

Dr. Benjamin Malzberg is now Director of the Bureau of Statistics, New York State Department of Mental Hygiene.

Dr. Henry B. Mann has taken a position as Research Associate at Ohio State University.

Dr. John W. Mauchly has been promoted to the rank of Assistant Professor of Electrical Engineering at the University of Pennsylvania.

Dr. R. v. Mises of Harvard University has been elected to membership in the American Academy of Arts and Sciences.

Frederick C. Mosteller is now Research Associate at Princeton University.

Franklin S. Nelson has been promoted from Assistant Statistician to Associate Statistician at Picatinny Arsenal.

Dr. Edward B. Olds, formerly Statistician with Curtiss-Wright, is now Economic Analyst with the United Nations Relief and Rehabilitation Administration.

Assistant Professor Charles K. Payne has been promoted to an associate professorship of mathematics at New York University.

Paul Peach, formerly with United States Rubber Company, is now Quality Control Consultant with the War Production Board.

Associate Professor Phillip J. Rulon has been promoted to the rank of Professor of Education and Acting Dean of the Harvard Graduate School of Education.

Assistant Professor Paul A. Samuelson has been promoted to the rank of Associate Professor of Economics at Massachusetts Institute of Technology.

Miss Marion M. Sandomire, formerly with the War Department, is now Statistician with the Navy Department, Bureau of Ships.

William C. Shelton, formerly with the Department of Commerce, is now Senior Mathematical Statistician with the Navy Department.

Blanche Skalak, formerly connected with Wilson Mechanical Instrument Company, is now Statistician with the Census Bureau.

Associate Professor L. Edwin Smart has been promoted to the rank of Professor of Economics at Ohio State University.

Dr. L. V. Toralballa, who has been a special instructor at the University of Michigan during the past year, is now at Princeton University for post-doctoral study.

Assistant Professor Abraham Wald has been promoted to Associate Professor of Statistics at Columbia University.

John F. Wyckoff has been promoted to Assistant Professor of Mathematics at Trinity College, Hartford.

Ruth Zwerling, formerly with the New York City Employees Retirement system, is now Research Assistant with the Milbank Memorial Fund.

New Members

The following persons have been elected to membership in the Institute:

Budne, Thomas A. M.A. (State Teachers Coll. Montclair) Asst. Prod. Mgr., Atlas Aircraft Prod. Corp., 405 E. 45 St., New York, N. Y. *475 Jersey St., Staten Island, N. Y.*

Darkow, Marguerite D. Ph.D. (Chicago) Asso. Prof. of Math., Hunter Coll., New York, N. Y. *16 E. 82 St., New York, 28, N. Y.*

Evers, Dillon Ph.D. (Iowa) Captain, Ord. Dept., Asst. Army Inspector of Ordnance, St. Louis Ord. Plant, 4300 Goodfellow Blvd., St. Louis, 20, Mo.

Hintermaier, John C. Grad. (Phil. Textile Inst.) Supervisory Chem. Forstmann Woolen Co., 2 Barbour Ave., Passaic, N. J. *21 Marilyn Place, Clifton, N. J.*

Katz, Amron H. B.A. (Wisconsin) Asso. Physicist, Chief, Shutter Unit, Aerial Photo Lab., TSELA-4D, Wright Field, Dayton, Ohio.

Lavin, Marvin M. M.S. (Chicago) Stat. Corning Glass Works. 80 E. Second St., Corning, New York.

Levine, Harriet A.B. (Hunter) Asst. Math. Stat., Stat. Res. Group, Columbia University, New York, N. Y. *309 W. 99th St., New York, 25, N. Y.*

Littauer, Sebastian B. D.Sc. (Mass. Inst. Tech.) Res. Eng., Bendix Radio Div., Bendix Aviation Corp., Towson, Md. *Box 6797, Towson, 4, Md.*

Pastore, Nicholas M.S. (C. C. N. Y.) Teacher, St. John's Prep. School, Brooklyn, N. Y. *2944 Radcliff Ave., New York, N. Y.*

Rodrigues, Milton da Silva Dr. (Brazil) Prof. of Stat., Univ. of Sao Paulo, Brazil. *International House, Rm. 323, 500 Riverside Drive, New York, 27, N. Y.*

Rothbard, Murray Senior (Columbia) 370 Central Park West, New York, N. Y.

Rubin, Ernest M.A. (Columbia) Res. Analyst, U. S. Dept. of Justice, Immigration & Nat. Service, 7104 Emerson Ave., Upper Darby, Pa.

Sousa, Alvaro Pedro de B.E. (Liverpool) Vice-Governor, Banco de Portugal, Monserrate, *Rua Infante de Sagres, Estoril, Portugal.*

Wirick, Grover C., Jr. A.B. (Mich State) Sec. Stat. Res. Dept., Milwaukee Community Fund & Counc. of Soc. Agencies. 3205 W. Wisconsin Ave., #C-8, Milwaukee, 8, Wis.

Report on the Third Meeting of the Pittsburgh Chapter of the Institute

The third meeting of the Pittsburgh Chapter of the Institute of Mathematical Statistics was held at Carnegie Union, Carnegie Institute of Technology, on Saturday, March 18, 1944. Twenty-five persons attended the meeting, including the following four members of the Institute:

G. G. Eldredge, H. J. Hand, F. G. Norris and E. M. Schrock.

At the morning session, Mr. E. M. Schrock, Aberdeen Proving Ground, Aberdeen, Md., presented a paper "Some Fundamental Aspects of the Shewhart Quality Control Chart". At the afternoon session, Mr. W. H. Thomas, National Tube Company Research Laboratory, presented an address "Analysis of Variance and Experimental Design in Metallurgical Research". Mr. F. G. Norris, President of the Pittsburgh Chapter, acted as chairman.

H. J. HAND,
Secretary-Treasurer of the Pittsburgh Chapter.

Report on the Fourth Meeting of the Pittsburgh Chapter of the Institute

The fourth meeting of the Pittsburgh Chapter of the Institute of Mathematical Statistics was held at Carnegie Union, Carnegie Institute of Technology on

Monday evening, October 2, 1944. Twenty-four persons attended the meeting, including the following four members of the Institute:

J. V. Sturtevant, J. Manuele, R. F. Passano, H. J. Hand.

Lieut. J. H. Curtiss, USNR, Bureau of Ships, Washington, D. C., presented a paper on the "Rational Sampling in the Preparation of Specifications".

R. F. Passano, Bethlehem Steel Company, Bethlehem, Pennsylvania, acted as chairman for the meeting.

H. J. HAND,
Secretary-Treasurer of the Pittsburgh Chapter.

DIRECTORY OF THE INSTITUTE OF MATHEMATICAL STATISTICS[1]

(As of November 1, 1944)

(The names of Fellows of the Institute are designated by *)

Acerboni, Prof. Argentino V. Dr. Ec. (Buenos Aires) Facultad de C. Economicas, Buenos Aires, Argentina. *Larroque 232, Banfield, Argentina.*

Allen, Roy G. D.Sc. (London) Combined Production and Resources Board, Soc. Security Bldg., Washington, D. C. *1305 Shepherd St., N.W., Washington, 11, D. C.*

Alt, Franz L. Ph.D. (Vienna) U. S. Army. *271 Fort Washington Ave., New York, 32, N. Y.*

Alter, Dinsmore Col., CAC, c/o A.T.S., Fort Mason, San Francisco, Calif.

Anderson, Paul H. Ph.D. (Illinois) Stat. Analyst, Distribution Mgt. Unit, Bur. of Foreign & Domestic Com., Washington, 25, D. C.

Anderson, Richard L. Ph.D. (Iowa State Coll.) Res. Math., Princeton Univ., Fine Hall, Princeton, N. J.

Anderson, Theodore W., Jr. M.A. (Princeton) Res. Math., Princeton Univ., Fine Hall, Princeton, N. J.

Andrews, T. Gaylord. Ph.D. (Nebraska) Instr. Barnard College, Columbia Univ., New York, N. Y.

Angell, Dorothy T. Stat. Analyst, Bell Tel. Labs., Murray Hill, N. J.

Arias, B., Jorge C.E. (Guatemala) 3 Avenida Sur 65, Guatemala City, Guatemala, Central America.

Arnold, Asso. Prof. Herbert E. Ph.D. (Yale) Wesleyan Univ., Middletown, Conn.

Arnold, Kenneth J. Ph.D. (Mass. Inst. Tech.) Sen. Math. Stat., Stat. Res. Group, Columbia Univ., 401 W. 118th St., New York, 27, N. Y.

Aroian, Leo A. Ph.D. (Michigan) Instr. Hunter Coll., New York, N. Y. *247 Wadsworth Ave., New York, 33, N. Y.*

Arrow, Kenneth J. M.A. (Columbia) 1st Lieut., A. C. Weather Div., Ac/As, O.C. & R., Hdqtrs. AAF, Pentagon Bldg., Washington, D. C. *1416 Nicholson St., N.W., Washington, 11, D. C.*

Bachelor, Robert W. M.B.A. (Washington) American Bankers Assn., 22 East 40th St., New York, 16, N. Y.

Bacon, Asso. Prof. Harold M. Ph.D. (Stanford) Stanford Univ., Stanford, Calif. *Box 1114.*

Bailey, Arthur L. B.S. (Michigan) Stat., American Mutual Alliance, 60 E. 42nd St., New York, N. Y. *P. O. Box 278, Ramsey, N. J.*

***Baker, Asst. Prof. George A.** Ph.D. (Illinois) Asst. Prof. of Math. and Asst. Stat. in the Exp. Sta., College of Agriculture, Univ. of California, Davis, Calif.

Baldwin, Woodson W. S.B. (Mass. Inst. Tech.) Capt., Ord. Dept., U. S. A., Office of Field Dir. of Ammunition Plants, 3629 Lindell Blvd., St. Louis, 8, Mo. *3745 Lindell Blvd.*

Bancroft, Asst. Prof. Theodore A. Ph.D. (Iowa State Coll.) Iowa State Coll., Math. Dept., Ames, Iowa.

Barnes, Jarvis M.A. (George Peabody Coll. for Teachers) Atlanta Board of Educ. 14th Floor, City Hall, Atlanta, Ga.

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[1] Members were asked to supply fresh information for this *Directory*. Records may be inexact or incomplete because of failure of some members to comply with this request. Changes in addresses, or errors in names, titles or addresses, should be reported to the Secretary.

Barnes, John L. Ph.D. (Princeton) Tech. Staff, Bell Tel. Labs., 463 West St., New York, N. Y. *19 E. River Rd., Rumson, N. J.*

Barr, Prof. Arvil S. Ph.D. (Wisconsin) Univ. of Wisconsin, Madison, Wis.

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